A posteriori error analysis of an augmented dual-mixed method in linear elasticity with mixed boundary conditions

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Abstract

We consider the augmented mixed finite element method introduced in [7] for the equations of plane linear elasticity with mixed boundary conditions. We develop an a posteriori error analysis based on the Ritz projection of the error and obtain an a posteriori error estimator that is reliable and efficient, but that involves a non-local term. Then, introducing an auxiliary function, we derive fully local reliable a posteriori error estimates that are locally efficient up to the elements that touch the Neumann boundary. We provide numerical experiments that illustrate the performance of the corresponding adaptive algorithm and support its use in practice.

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1 Introduction

Recently, a new augmented mixed finite element method was introduced in [7] for the linear elasticity problem in the plane with homogeneous Dirichlet and mixed boundary conditions. The case of non-homogeneous Dirichlet boundary conditions was later analyzed in [8] and the corresponding 3D version can be found in [9].

The approach in [7] relies on the mixed method of Hellinger and Reissner, that is enriched with suitable least-squares type terms arising from the equilibrium equation, the constitutive law and the relation that defines the rotation in terms of the displacement. In the case of mixed boundary conditions, the Neumann boundary condition is imposed weakly, through the use of a Lagrange multiplier that can be interpreted as the trace of the displacement on the Neumann boundary. That method allows to use Raviart-Thomas elements of the lowest order to approximate the stress tensor, continuous

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piecewise linear elements to approximate the displacement and piecewise constants to approximate the rotation; the Lagrange multiplier on the Neumann boundary can be approximated by continuous piecewise linear elements on a suitable partition of that boundary, as we will see later. The resulting discrete scheme is well-posed and free of locking for appropriate values of the stabilization parameters.

Concerning the a posteriori error analysis, an a posteriori error estimator of residual type was derived in [2] in the case of pure homogeneous Dirichlet boundary conditions. More recently, we extended that analysis to the case of non-homogeneous Dirichlet and mixed boundary conditions in [3]. Although this a posteriori error estimator is reliable and efficient, its computation could be expensive, specially if one thinks in its extension to the three-dimensional case.

In this paper, we consider the augmented dual-mixed method introduced in [7] in the case of mixed boundary conditions and develop an a posteriori error analysis based on the Ritz projection of the error. We obtain an a posteriori error estimator that is reliable and efficient, but that contains a non-local term. We then introduce an auxiliary function and derive fully local a posteriori error estimates that are reliable and locally efficient up to those elements that touch the Neumann boundary (see Theorem 5 below). As compared with the a posteriori error estimator introduced in [3] in the case of mixed boundary conditions, the a posteriori error estimates presented here do not involve tangential nor normal jumps. So, from a practical point of view, they are cheaper and easier to implement. Moreover, numerical experiments support the use of the new a posteriori error estimates in practice. We also remark that the present approach can be easily extended to the three-dimensional case.

The rest of the paper is organized as follows. In Section 2, we recall the augmented variational formulation proposed in [7] for the linear elasticity problem in the plane with mixed boundary conditions, the corresponding Galerkin scheme and the simplest finite element subspaces that can be used. In Section 3, we develop the a posteriori error analysis and propose the new a posteriori error estimates. Finally, in Section 4 we provide several numerical experiments that support the use of the new a posteriori error estimates in practice.

We end this section with some notations to be used throughout the paper. Given a Hilbert space H, we denote by H^2 (resp., $H^{2\times 2}$) the space of vectors (resp., square tensors) of order 2 with entries in H. Given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we denote $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\operatorname{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. We also use the standard notations for Sobolev spaces and norms. Finally, C or c (with or without subscripts) denote generic constants, independent of the discretization parameters, that may take different values at different occurrences.

2 The augmented mixed finite element method

In this section we recall the augmented mixed finite element method introduced in [7] to solve the linear elasticity problem in the plane with mixed boundary conditions. Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with a Lipschitz-continuous boundary Γ, and let Γ_D and Γ_N be two disjoint open subsets of Γ such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and Γ_D has positive measure. We denote by $\mathcal{C}$ the elasticity operator determined by Hooke's law, that is,

$$\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\operatorname{tr}(\boldsymbol{\zeta})\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta}\,, \qquad \forall\,\boldsymbol{\zeta} \,\in\, [L^2(\Omega)]^{2\times 2}\,,$$