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RESEARCH ARTICLES

C-Algebras on some Free-Banach Spaces*

J. Aguayo[1], M. Nova[2***], and J. Ojeda[1****]**

[1] *Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción-Chile*

[2] *Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción-Chile*

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Abstract—The main goal of this work is to study the Gelfand spaces of some commutative Banach algebras with unit within the space of bounded linear operators. We will also show, under special condition, that this algebra is isometrically isomorphic to some space of continuous functions defined over a compact. Such isometries preserve idempotent elements. This fact will allow us to define the respective associated measure which is known as spectral measure. Let us also notice that this measure is obtained by restriction of the reciprocal of the Gelfand transform to the set of characteristic functions of clopen subsets of the spectrum of above algebra. We will finish this work showing that each element of such algebras described above can be represented as an integral of some continuous function, where the integral has been defined through the spectral measure.

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1. INTRODUCTION AND NOTATIONS

Many researchers have tried to generalize the elemental studies of Banach algebras from classical case to vectorial structures over non-archimedean fields. The first big task was to find a result similar to the Gelfand-Mazur Theorem in this context. But, this theorem failed since every field $\mathbb{K}$ with a "non-archimedean valuation" is contained in another field $\widetilde{\mathbb{K}}$ whose valuation is an extension of previous one and both fields are different.

One of the pioneers in the study of non-archimedean Banach algebras of linear operators and spectral theory in this context has been M. Vishik [9], especially in the class of linear operators which admit compact spectrum. We can also mention another important authors as V. Berkovich [4], who made a deep study of this subject on his survey, as well as S. Ludkovsky and B. Diarra [6], who developed the spectral integration for non-archimedean Banach spaces.

The main goal of this work is to study the Gelfand space of some commutative Banach algebras with unit within the space of bounded linear operators. We will also show, under special conditions, that each of these algebras is isometrically isomorphic to some space of continuous functions defined over a compact. Such isometries preserve idempotent elements. This fact will allow us to define the respective associated measure which is known as spectral measure. We will finish this work showing that each element of such algebras described above can be represented as an integral of some continuous function, where the integral has been defined through the spectral measure.

Throughout this paper, $\mathbb{K}$ denotes a complete, non-archimedean valued field and its residue class field is formally real.

*The text was submitted by the authors in English.

**E-mail: jaguayo@udec.cl

***E-mail: mnova@ucsc.cl

****E-mail: jacqojeda@udec.cl

Summable families will be used in this work and, therefore, we will recall the definition of such concept. We know that if E is a normed space and $(x_\alpha)_{\alpha\in\Gamma}$ is a family of elements of E, then we say that the family $(x_\alpha)_{\alpha\in\Gamma}$ is summable with sum $x \in E$ if for every $\epsilon > 0$ there is a finite subset Λ_ϵ of Γ such that, for all finite subsets Λ with $\Lambda_\epsilon \subset \Lambda \subset \Gamma$, we have

$$\left\| x - \sum_{\gamma\in\Lambda} x_\gamma \right\| < \epsilon.$$

In this case, we write $x = \sum_{\alpha\in\Gamma} x_\alpha$.

We also know that:

1. if the family $(x_\alpha)_{\alpha\in\Gamma}$ is summable, then $\lim_{\alpha\in\Gamma} x_\alpha = 0$ in the sense that for every $\epsilon > 0$, the set $\{\alpha \in \Gamma : |x_\alpha| \geq \epsilon\}$ is finite;

2. the converse is also true if E is a Banach space;

3. if the family $(x_\alpha)_{\alpha\in\Gamma}$ is summable, then $x_\alpha = 0$ except for a countable subset of Γ; hence $x = \sum_{\alpha\in\Gamma} x_\alpha = \sum_{k\in\mathbb{N}} x_{\alpha_k}$.

A non-archimedean Banach space E is said to be a Free Banach space if there exists a family $\{e_\zeta\}_{\zeta\in\Upsilon}$ of non-null vectors of E such that any element x of E can be written in the form of convergent sum $x = \sum_{\zeta\in\Upsilon} x_\zeta e_\zeta,\ x_\zeta \in \mathbb{K}$, and $\|x\| = \sup_{\zeta\in\Upsilon} |x_\zeta| \, \|e_\zeta\|$. The family $\{e_\zeta\}_{\zeta\in\Upsilon}$ is called orthogonal basis of E. It is known that $T \in \mathcal{L}(E)$ can be written as $T = \sum_{\zeta,\xi\in\Upsilon} \alpha_{\zeta\xi} e'_\xi \otimes e_\zeta$, where $\lim_{\zeta\in\Upsilon} |\alpha_{\zeta\xi}| \, \|e_\zeta\| = 0$ $\forall \xi \in \Upsilon$ and e'_ξ is the element of the dual space E' of E such that $e'_\xi(e_\zeta) = \delta_{\xi\zeta}$ (Kronecker's delta) and the operator $e'_\xi \otimes e_\zeta$ is defined by $e'_\xi \otimes e_\zeta(x) = e'_\xi(x)\, e_\zeta = x_\xi e_\zeta$.

If $s : \Upsilon \to (0, \infty)$, then an example of Free Banach space is $c_0(\Upsilon, \mathbb{K}, s)$, the collection of all $x = (x_\zeta)_{\zeta\in\Upsilon}$ such that $\lim_{\zeta\in\Upsilon} |x_\zeta|\, s(\zeta) = 0$ and $\|x\| = \sup_{\zeta\in\Upsilon} |x_\zeta|\, s(\zeta)$. For more details concerning Free Banach spaces, we refer the reader to [5]

Let I be an arbitrary set of indices and let $c_0(I)$ be the Free Banach space $c_0(I, \mathbb{K}, s)$, when $s \equiv 1$, i.e.

$$c_0(I) := \left\{ (x_i)_{i\in I} : x_i \in \mathbb{K};\ \lim_{i\in I} x_i = 0 \right\}.$$

Its norm is given by

$$\|x\|_\infty := \sup_{i\in I} |x_i| \, , \text{ with } x = (x_i)_{i\in I} \in c_0(I) \, .$$

The bilinear form $\langle \cdot, \cdot \rangle : c_0(I) \times c_0(I) \to \mathbb{K}$ defined by

$$\langle x, y \rangle = \sum_{i\in I} x_i y_i$$

is an inner product in the sense of [7] since the residue class field of $\mathbb{K}$ is formally real. Even more,

$$\|x\|_\infty^2 = |\langle x, x \rangle| \, .$$

On the other hand, if we define the operation

$$\lambda \cdot \mu = (\lambda_i \mu_i)_{i\in I}\, ;\ \ \lambda, \mu \in c_0(I),$$

then $c_0(I)$ is a commutative Banach algebra without unity.

As we know, $c_0(I)^+ := \mathbb{K} \oplus c_0(I)$ with the operations

$$(\alpha, \lambda) + (\beta, \mu) = (\alpha + \beta, \lambda + \mu)$$

$$(\alpha, \lambda) \cdot (\beta, \mu) = (\alpha\beta, \beta\lambda + \alpha\mu + \lambda \cdot \mu) \, ,$$