RESEARCH ARTICLE

A stabilized mixed method applied to Stokes system with nonhomogeneous source terms: The stationary case

Dedicated to Prof. R. Rodríguez, on the occasion of his 65th birthday

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Summary
This article is concerned with the Stokes system with nonhomogeneous source terms and nonhomogeneous Dirichlet boundary condition. First, we reformulate the problem in its dual mixed form, and then, we study its corresponding well-posedness. Next, in order to circumvent the well-known Babuška-Brezzi condition, we analyze a stabilized formulation of the resulting approach. Additionally, we endow the scheme with an a posteriori error estimator that is reliable and efficient. Finally, we provide numerical experiments that illustrate the performance of the corresponding adaptive algorithm and support its use in practice.

KEYWORDS
a posteriori error estimates, augmented mixed formulation, Ritz projection of the error

1 | INTRODUCTION

In the work of Cai et al,1 a dual mixed finite element method for the incompressible fluid flow was introduced and analyzed. The approach there follows the ideas developed in the other work of Cai et al,2 ie, the incompressible fluid flow is reformulated using the new variable so-called pseudostress, which is in relation with the pressure and gradient of the velocity. The main advantage of this new variable is the accurate approximation to physical quantities such as the stress and vorticity, allowing to use the pair of conforming Raviart-Thomas with discontinuous polynomial as the finite element space. Furthermore, in order to obtain more flexibility in the finite element spaces, the stabilization of this approach has been studied in the work of Figueroa et al.3 In addition, its corresponding extension to quasi Newtonian flows and Brinkman model were developed in the works of Gatica et al4 and Barrios et al,5 respectively.

On the other hand, concerning linear elasticity problem, in the work of Barrios et al,6 the authors present an alternative a posteriori error estimator to the previous one developed in a former work.7 This approach is based on the Ritz projection...
of the error (see the work of Barrios and Gatica). As result, in the case of homogeneous Dirichlet boundary condition, we obtain a reliable and local efficient a posteriori error estimator that only requires the computation of four residuals per element, which has a low computational cost comparing with the eleven terms included in the estimator developed in the work of Barrios et al.7 for the same case. This kind of a posteriori error estimator, at least, has been developed satisfactorily in different directions, for example, the Poisson problem, Darcy flow, the Brinkman model, linear elasticity, and recently the Oseen equations.12

Then, our interest in this article is to study the Stokes system with nonhomogeneous source terms, using a stabilized mixed approach. In order to describe as clear as possible the stabilization procedure, we begin by applying a dual mixed approach, where the well-posedness is consequence of the standard Babuška-Brezzi theory. After that, and as the first novelty of this article, we include the a priori error analysis of the stabilized formulation for this kind of problems, which allows us to expand the choice of stable pairs that could be used to approximate the solution. In addition, and strongly motivated by the reduction of computational cost obtained with the a posteriori error estimator based on Ritz projection of the error, we endow the new approach with an estimator of this type, which consists of only five terms and is reliable and efficient. These features constitute another novelty of the current work.

In what follows, in order to describe the model of interest, we let Ω be a bounded and simply connected domain in \( \mathbb{R}^2 \) with polygonal boundary \( \Gamma \). Then, given the source terms \( f \in L^2_0(\Omega), f \in [L^2(\Omega)]^2 \) and \( g \in [H^{1/2}(\Gamma)]^2 \), we look for the velocity (vector field) \( u \) and the pressure (scalar field) \( p \) such that

\[
-\nu \Delta u + \nabla p = f \quad \text{in} \quad \Omega, \quad \text{div}(u) = \frac{1}{\nu} f \quad \text{in} \quad \Omega, \quad \text{and} \quad u = g \quad \text{on} \quad \Gamma.
\] (1)

where \( \nu > 0 \) is the fluid viscosity of the flow, which will be assumed to be constant, and the datum \( g \) satisfies the compatibility condition \( \int g \cdot n = 0 \), with \( n \) being the unit outward normal at \( \Gamma \). In addition, for uniqueness purposes, we seek \( p \in L^2_0(\Omega) := \{ q \in L^2(\Omega) : \int_\Omega q = 0 \} \).

The rest of this paper is organized as follows. In Section 2, we analyze the dual mixed variational formulation obtained from the model flow (1), including the corresponding Galerkin scheme with the finite element subspaces of lowest degree that can be used. Section 3 is concerned with a stabilization of the dual mixed approach, whereas in Section 4, we develop an a posteriori error analysis and deduce a new a posteriori error estimator. Finally, in Section 5, we provide several numerical experiments that support the use of our a posteriori error estimator in practice.

We end this section with some notations to be used throughout this paper. Given a Hilbert space \( H \), we denote by \( H^2 \) (respectively, \( H^{2x2} \)) the space of vectors (respectively, square tensors of order 2) with entries in \( H \). Given \( \tau := (\tau_{ij}) \) and \( \zeta := (\zeta_{ij}) \in \mathbb{R}^{2x2} \), we set \( \tau^t := (\tau_{ji}) \), \( \text{tr}(\tau) := \tau_{11} + \tau_{22} \), \( \tau : \zeta := \sum_{i,j=1}^2 \tau_{ij} \zeta_{ij} \). Moreover, we introduce the deviator of \( \tau \) by \( \tau^d := \tau - \frac{1}{2} \text{tr}(\tau) I \), where \( I \in \mathbb{R}^{2x2} \) denotes the identity tensor. We also use the standard notations for Sobolev spaces and norms. Finally, \( C \) or \( c \) (with or without subscripts) denote generic constants, independent of the discretization parameters, that may take different values at different occurrences.

2 \ THE DUAL MIXED FORMULATION

We begin this work by introducing the dual mixed formulation for the Stokes system. To this end, we first reformulate problem (1) introducing the pseudostress \( \sigma := \nu \nabla u - p I \) in \( \Omega \) as an additional unknown. Since \( \text{div} (u) = \frac{1}{\nu} f \) in \( \Omega \), it is not difficult to deduce that \( p = \frac{1}{\nu} f - \frac{1}{2} \text{tr}(\sigma) \) in \( \Omega \), which implies that \( \sigma \in H_0 := \{ \tau \in H(\text{div}; \Omega) : \int_\Omega \text{tr}(\tau) = 0 \} \). This relation allows us to eliminate the pressure of the second order problem (1) and thus derive the following first order system: Find \( (\sigma, u) \in H_0 \times [H^1(\Omega)]^2 \) such that

\[
\frac{1}{\nu} \sigma - \nabla u = -\frac{1}{2} f I \quad \text{in} \quad \Omega, \quad \text{div}(\sigma) = -f \quad \text{in} \quad \Omega, \quad \text{and} \quad u = g \quad \text{on} \quad \Gamma.
\] (2)

Proceeding in the usual way, we deduce the variational mixed formulation based on velocity and pseudostress, which reads as follows: Find \( (\sigma, u) \in H_0 \times [L^2(\Omega)]^2 \) such that

\[
a(\sigma, \tau) + b(\tau, u) = G(\tau) \quad \forall \tau \in H_0, \quad \text{(3)}
\]

\[
b(\sigma, v) = F(v) \quad \forall v \in [L^2(\Omega)]^2, \quad \text{(4)}
\]