Principal Poincaré Pontryagin Function associated to some families of Morse real polynomials

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Abstract

It is known that the Principal Poincaré Pontryagin Function is generically an Abelian integral. We give a sufficient condition on monodromy to ensure that it is an Abelian integral also in non generic cases.

In non generic cases it is an iterated integral. Uribe [17, 18] gives in a special case a precise description of the Principal Poincaré Pontryagin Function, an iterated integral of length at most 2, involving logarithmic functions with only one ramification at a point at infinity. We extend this result to some non isodromic families of real Morse polynomials.

Keywords: Perturbation, First return map, Iterated integrals, Monodromy, Stratification.

MSC: 34M35; 34C08;14D05

1 Introduction

Throughout the paper $F$ denotes a Morse polynomial $F(x, y) : \mathbb{C}^2 \to \mathbb{C}$ with real coefficients, of degree $d \geq 3$ and with $d$ distinct real points at infinity. It always has $(d - 1)^2$ critical points but in non generic cases it can have less than $(d - 1)^2$ critical values. The one-form $dF$ defines a foliation of $\mathbb{C}^2$. We consider a family of ovals $\delta(t)$ in regular fibers $F = t$ for $t$ in some open interval and a transverse section to these ovals, parametrized by

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Let $\omega(x,y)$ be a real polynomial one-form. To a one-parameter foliation defined by the perturbation $dF + \varepsilon \omega$ for a small parameter $\varepsilon$ is associated the displacement map which is the difference of the first return map and the identity. It is analytic with respect to $\varepsilon$. The family of ovals is destroyed if and only if the expansion with respect to $\varepsilon$ of the displacement map is not identically 0. In order to control the number of isolated zeroes of the displacement map, or in other words the number of limit cycles of the perturbed foliation, it is crucial to know the nature of the first nonzero coefficient of its expansion in $\varepsilon$. It is called the Generating Function in [10]. Following [8] we will call it the Principal Poincaré Pontryagin Function.

It is known that it is an iterated integral [5] and that its length depends on the monodromy group of the Milnor fibration associated to the non-perturbed polynomial $F$ [10]. Generically, that is if all $(d - 1)^2$ critical values are distinct, this monodromy acts transitively on the homology with complex coefficients of regular fibers $F^{-1}(t)$ and the Principal Poincaré Pontryagin Function is an Abelian integral. It may also be an Abelian integral in non-generic cases [12, 13]. If it is not an Abelian integral, the simplest case is the one where it is a length 2 iterated integral, for example if $F$ is a triangle [11], or more generally if $F$ is the product of $d$ linear factors, $F = \ell_1 \cdots \ell_d$, satisfying to the following Hypothesis [17, 18].

**Hypothesis 1** The $d$ points at infinity $\ell_k = 0$ are distinct, all critical points are Morse points, and the 0-level is the only critical level containing more than one critical point. The $d(d - 1)/2$ intersection points of the line $\ell_k = 0$ are real.

These properties ensure that the 0-level of $F$ is what A’Campo calls a divide in [1, 2], see Section 3 for the definition. We keep this terminology.

**Definition 1** A polynomial $F = \ell_1 \cdots \ell_d$ is a generic divide in lines if it satisfies to Hypothesis 1.

In [18] one of us proves that for generic divides in lines the Principal Poincaré Pontryagin Function is an iterated integral of length at most 2. The proof uses monodromy properties of divides. The divide shows all the homology of regular fibers $F^{-1}(t)$ and allows also to compute the monodromy. Since Hypothesis 1 is stable it is natural to hope that the Principal Poincaré Pontryagin Function remains a length 2 iterated integral after a small perturbation. In Section 2 we give two examples of one-parameter small perturbations of generic divides in lines and we check that the Principal Poincaré Pontryagin...