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Low cost a posteriori error estimators for an augmented mixed FEM in linear elasticity

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Dedicated to Professor Rodolfo Rodríguez on the occasion of his 60th birthday.

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ABSTRACT

We consider an augmented mixed finite element method applied to the linear elasticity problem and derive a posteriori error estimators that are simpler and easier to implement than the ones available in the literature. In the case of homogeneous Dirichlet boundary conditions, the new a posteriori error estimator is reliable and locally efficient, whereas for non-homogeneous Dirichlet boundary conditions, we derive an a posteriori error estimator that is reliable and satisfies a *quasi-efficiency* bound. Numerical experiments illustrate the performance of the corresponding adaptive algorithms and support the theoretical results.

1. Introduction

In this paper, we consider the augmented dual-mixed method introduced in [10,11] for the linear elasticity problem in the plane with Dirichlet boundary conditions and extended in [12] to the three-dimensional case. The approach in [10–12] relies on the mixed method of Hellinger and Reissner, that provides simultaneous approximations of the displacement $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$. The symmetry of $\boldsymbol{\sigma}$ is imposed weakly, through the use of a Lagrange multiplier that can be interpreted as the rotation $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{t})$. Then, suitable Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and from the relation that defines the rotation in terms of the displacement are added. Besides, in the case of non-homogeneous Dirichlet boundary conditions, the bilinear form is augmented with a consistency term related with the boundary condition. The resulting augmented variational formulation is coercive in the whole space for appropriate values of the stabilization parameters, with a coercivity constant independent of the Lamé parameter λ. Therefore, the associated Galerkin scheme is well-posed and free of locking for *any* choice of finite element subspaces, which is in turn the main advantage of this method.

On the other hand, the use of adaptive algorithms based on a posteriori error estimates guarantees good convergence behavior of the finite element solution of a boundary value problem. Several a posteriori error estimators are already available in the literature for the usual mixed finite element method in linear elasticity (see [5,7,15,8,14,9]). Concerning the a posteriori error analysis of the augmented scheme presented in [10] in the case of pure homogeneous Dirichlet boundary

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conditions, an a posteriori error estimator of residual type was introduced in [3]. That analysis was extended recently to the cases of pure non-homogeneous Dirichlet boundary conditions and mixed boundary conditions with non-homogeneous Neumann data; cf. [4]. The a posteriori error estimators derived in [3] and [4] are both reliable and efficient, and involve the computation of eleven residuals per element in the homogeneous case, and thirteen residuals per element in the non-homogeneous one; both include normal and tangential jumps.

In this paper, we present new a posteriori error estimators for the augmented dual-mixed methods proposed in [10–12] in the case of Dirichlet boundary conditions. The analysis is based on the use of a projection of the error and the homogeneous and non-homogeneous cases are studied separately. In the case of homogeneous boundary conditions, we obtain an a posteriori error estimator that is reliable, locally efficient and only requires the computation of four residuals per element. Moreover, this a posteriori error estimator is the first one derived for the augmented method proposed in [12]. When non-homogeneous boundary conditions are imposed, we derive two new reliable a posteriori error estimators, one valid in 2D and 3D, and a second one that is only valid in 2D. The latter is locally efficient in the elements that do not touch the boundary and requires the computation of four residuals per element in the interior triangles, five residuals per element in the triangles with exactly one node on the boundary and six residuals per element in the triangles with a side on the boundary. Neither of these a posteriori error estimators require the computation of normal nor tangential jumps, which makes them easy to implement.

The rest of the paper is organized as follows. In Section 2 we recall the main features of the augmented dual-mixed method introduced in [10,12] for the linear elasticity problem with homogeneous Dirichlet boundary conditions. Then, we use the Ritz projection of the error to derive the new a posteriori error estimator and show that it is reliable and locally efficient. The extension to the case of non-homogeneous Dirichlet boundary conditions is developed in Section 3, where we first recall the dual-mixed method from [11,12]. Finally, in Section 4 we provide several numerical experiments that illustrate the performance of the corresponding adaptive algorithms and support the theoretical results.

In what follows, we will use the standard notations for Sobolev spaces and norms. We let $\Omega \subset \mathrm{R}^d$ $(d = 2, 3)$ be a bounded domain with a Lipschitz-continuous boundary Γ. We denote $H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{d\times d} : \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^d\}$, $H_0 := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\}$ and $[L^2(\Omega)]^{d\times d}_{\mathtt{skew}} := \{\boldsymbol{\eta} \in [L^2(\Omega)]^{d\times d} : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\}$. The duality pairing between $[H^{-1/2}(\Gamma)]^d$ and $[H^{1/2}(\Gamma)]^d$ with respect to the $[L^2(\Gamma)]^d$-inner product is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameter, that may take different values at different occurrences.

2. Homogeneous Dirichlet boundary conditions

Let $\mathbf{f} \in [L^2(\Omega)]^d$ be a given volume force. We denote by C the elasticity operator determined by Hooke's law, that is,

$$\mathrm{C}\boldsymbol{\zeta} := \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\mathbf{I} + 2\mu\boldsymbol{\zeta}, \quad \forall \boldsymbol{\zeta} \in \left[L^2(\Omega)\right]^{d\times d}, \tag{1}$$

where $\lambda, \mu > 0$ are the Lamé parameters and $\mathbf{I}$ is the identity matrix in $\mathrm{R}^{d\times d}$. The problem of linear elasticity with homogeneous Dirichlet boundary conditions consists in finding the displacement $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$ such that

$$\begin{cases} -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} & \text{in } \Omega, \\ \boldsymbol{\sigma} = \mathrm{C}\boldsymbol{\varepsilon}(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}})$ is the strain tensor of small deformations. In the next subsection, we recall the augmented dual-mixed method proposed in [10,12] to solve problem (2).

2.1. The augmented dual-mixed finite element method

Let κ_1, κ_2 and κ_3 be positive parameters. We denote $\mathbf{H} := H_0 \times [H^1_0(\Omega)]^d \times [L^2(\Omega)]^{d\times d}_{\mathtt{skew}}$ and $\tilde{\mathbf{H}} := H_0 \times [H^1(\Omega)]^d \times [L^2(\Omega)]^{d\times d}_{\mathtt{skew}}$. We define the bilinear form $A : \tilde{\mathbf{H}} \times \tilde{\mathbf{H}} \to \mathrm{R}$ and the linear functional $F : \tilde{\mathbf{H}} \to \mathrm{R}$ as follows:

$$\begin{aligned} A\big((\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})\big) := {} & \int_\Omega \mathrm{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\tau} : \boldsymbol{\gamma} \\ & - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} + \kappa_1 \int_\Omega \big(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathrm{C}^{-1}\boldsymbol{\sigma}\big) : \big(\boldsymbol{\varepsilon}(\mathbf{v}) + \mathrm{C}^{-1}\boldsymbol{\tau}\big) \\ & + \kappa_3 \int_\Omega \Big(\boldsymbol{\gamma} - \frac{1}{2}\big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\big)\Big) : \Big(\boldsymbol{\eta} + \frac{1}{2}\big(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\big)\Big) \\ & + \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}), \quad \forall (\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) \in \tilde{\mathbf{H}}, \end{aligned} \tag{3}$$