AN ADAPTIVE RESIDUAL LOCAL PROJECTION FINITE ELEMENT METHOD FOR THE NAVIER–STOKES EQUATIONS

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ABSTRACT. This work proposes and analyses an adaptive finite element scheme for the fully non-linear incompressible Navier-Stokes equations. A residual a posteriori error estimator is shown to be effective and reliable with respect to the natural norms. The error estimator relies on a Residual Local Projection (RELP) finite element method for which we prove well-posedness under mild conditions. Several well-established numerical tests assess the theoretical results.

1. INTRODUCTION

A posteriori error analysis for adaptive finite element methods has been a very active and successful subject of research since the pioneering work of Babuska and Rheinboldt in [8]. In the context of fluid flow problems, researchers have been focused on improving numerical precision while making the computational cost affordable. For the Stokes problem we cite the relevant works by Verfürth [31], Bank and Welfert [9] and Ainsworth and Oden [2]. Regarding the Navier-Stokes equations it is worth mentioning the residual-based estimators proposed in [7, 13, 17, 22], the goal-oriented scheme in [12], and the hierarchical a posteriori error estimator in [5] and the ones based on local problem solutions in [21, 25] (see also [1, 33] for an overview).

Stabilized finite element methods for Navier-Stokes equations use equal-order pairs of interpolation spaces for the velocity and pressure. Well-balanced numerical diffusion may be also incorporated into such methods through the stabilization parameter. This is a crucial point when it comes to numerically solving advection dominated (high Reynolds number) flows (see [18, 29] or [14], for instance). The association of stabilized methods with a posteriori error estimators greatly improves the quality of the numerical solutions while keeping the computational cost relatively low (see [3]). Such a feature is particularly attractive if one approximates solutions with multiple scales, as in the case of the non-linear Navier-Stokes equations.

Residual Local Projection (RELP) stabilized methods add new stabilization to the Galerkin method as a result of a space enriching strategy. First proposed in [10, 11] for the Stokes operator, and further extended to the fully non-linear Navier-Stokes equations in [4], these methods rely on the solution of element-wise problems. Such a local solution designs the stabilization parameter with the right dose of numerical diffusion and stabilizes the equal order and the simplest elements. In this work, we develop a new residual-based a posteriori error estimator for the non-linear incompressible Navier-Stokes equations. To this end, we consider a variation of the RELP method proposed in [4] for which we prove the existence and the uniqueness of the

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solution. Also, we prove that the new estimator is effective and reliable following closely the theory presented in Verfürth [32].

The paper is organized as follows: Section 2 states the problem and introduces preliminary results. Section 3 presents the RELP method and the proof of the existence of a solution. The residual a posteriori error estimator is analyzed in Section 4, followed by numerical validations in Section 5. Finally, concluding remarks are given in Section 6 and the appendix includes the proof of a local unique solution for the RELP method.

2. Model problem and Preliminary results

The steady incompressible Navier–Stokes problem consists of finding the velocity $\boldsymbol{u}$ and the pressure p solution of

$$\text{(NS)} \quad \begin{cases} -\nu\,\Delta \boldsymbol{u} + (\nabla \boldsymbol{u})\,\boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \nabla\cdot \boldsymbol{u} &= 0 \qquad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^2$ is a polygonal open domain, $\nu \in \mathbb{R}^+$ is the fluid viscosity and $\boldsymbol{f} \in L^2(\Omega)^2$ is a given function. We set $\mathbf{V} := H_0^1(\Omega)^2$ and $Q := L_0^2(\Omega)$ and introduce the weak form of (NS): *Find* $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ *such that*

$$\nu\,(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\nabla \boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot \boldsymbol{v}) - (q, \nabla\cdot \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } (\boldsymbol{v}, q) \in \mathbf{V} \times Q, \tag{1}$$

here $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product, where we use the same notation for vector, or tensor, valued functions.

Problem (1) may be rewrite in a more convenient form in view of analysis. To this end, consider the operator $F : \mathbf{V} \times Q \longrightarrow (\mathbf{V} \times Q)'$ defined by

$$\langle F(\boldsymbol{u}, p), (\boldsymbol{v}, q)\rangle := \nu\,(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\nabla \boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot \boldsymbol{v}) - (q, \nabla\cdot \boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{v}),$$

where $\langle\cdot,\cdot\rangle$ is the duality product in $(\mathbf{V}\times Q)' \times (\mathbf{V}\times Q)$. Note that (1) is equivalent to: *Find* $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ *such that*

$$\langle F(\boldsymbol{u}, p), (\boldsymbol{v}, q)\rangle = 0 \qquad \forall (\boldsymbol{v}, q) \in \mathbf{V} \times Q. \tag{2}$$

To present the discrete version of (2) and the numerical analysis of it, we need some notations and also some standard technical results. We denote the derivative of F with respect to $(\boldsymbol{u}, q)$ at $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$ by $D_{\boldsymbol{u},p}F(\boldsymbol{v}, q) \in \mathcal{L}(\mathbf{V} \times Q)$, where $\mathcal{L}(\mathbf{V} \times Q)$ stands for the space of bounded linear mappings acting on elements of $\mathbf{V} \times Q$ with values in $\mathbf{V} \times Q$ and equipped with the norm $\|\cdot\|_{\mathcal{L}(\mathbf{V}\times Q)}$ with its usual meaning.

We assume that problem (2) has a solution $(\boldsymbol{u}, p)$ and $D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)$ is an isomorphism from $\mathbf{V} \times Q$ onto $(\mathbf{V} \times Q)'$ (see Section IV.3.1 in [20]). Also, we assume that there is a constant $R_0 > 0$ such that $(\boldsymbol{u}, p)$ is unique in the ball $\mathbb{B}((\boldsymbol{u}, p), R_0)$ (see Section IV.3.2 in [20]). Thereby, the differential operator $D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)$ is Lipschitz continuous at $(\boldsymbol{u}, p)$, i.e.,

$$\gamma := \sup_{(\boldsymbol{v},q)\in\mathbb{B}((\boldsymbol{u},p),R_0)} \frac{\|D_{\boldsymbol{u},p}F(\boldsymbol{v}, q) - D_{\boldsymbol{u},p}F(\boldsymbol{u}, p)\|_{\mathcal{L}((\mathbf{V}\times Q),(\mathbf{V}\times Q)')}}{\|(\boldsymbol{v} - \boldsymbol{u}, q - p)\|_{\mathbf{V}\times Q}} < \infty.$$

We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family of triangulations of Ω into triangles K with boundary ∂K and diameter $h_K := diam(K)$, and $h := \max\{h_K : K \in \mathcal{T}_h\}$. The set of internal edges F reads $\mathcal{E}_h$ and we define $h_F := |F|$. We denote by $\boldsymbol{n}$ the outward normal vector on ∂K; by $[\![\,\cdot\,]\!]_F$ we mean the jump of v over