A generalized action for $(2+1)$-dimensional Chern–Simons gravity

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Abstract

We show that the so-called semi-simple extended Poincaré (SSEP) algebra in $D$ dimensions can be obtained from the anti-de Sitter algebra $so(D-1,2)$ by means of the $S$-expansion procedure with an appropriate semigroup $S$. A general prescription is given for computing Casimir operators for $S$-expanded algebras, and the method is exemplified for the SSEP algebra. The $S$-expansion method also allows us to extract the corresponding invariant tensor for the SSEP algebra, which is a key ingredient in the construction of a generalized action for Chern–Simons gravity in $2+1$ dimensions.

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1. INTRODUCTION

In Refs. [1–4], the Poincaré algebra of rotations $J_{ab}$ and translations $P_a$ in $D$-dimensional spacetime has been extended by the inclusion of the second-rank tensor generator $Z_{ab}$ in the following way:

\[ [J_{ab}, J_{cd}] = \eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \]  
(1)

\[ [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \]  
(2)

\[ [P_a, P_b] = c Z_{ab}, \]  
(3)

\[ [J_{ab}, Z_{cd}] = \eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}, \]  
(4)

\[ [Z_{ab}, P_c] = \frac{4a^2}{c} \left( \eta_{bc} P_a - \eta_{ac} P_b \right), \]  
(5)

\[ [Z_{ab}, Z_{cd}] = \frac{4a^2}{c} \left[ \eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} \right], \]  
(6)

where $a$ and $c$ are constants. It is remarkable that the Lie algebra (1)–(6) is semi-simple, in contrast to the Poincaré and extended Poincaré algebras [cf. eqs. (1.1) and (1.2) of Ref. [3]]. Note that, in the $a \to 0$ limit, the algebra (1)–(6) reduces to the algebra in eq. (1.2) of Ref. [3]. The semi-simple extended Poincaré (SSEP) algebra (1)–(6) can be rewritten in the form

\[ [N_{ab}, N_{cd}] = \eta_{ad} N_{bc} + \eta_{bc} N_{ad} - \eta_{ac} N_{bd} - \eta_{bd} N_{ac}, \]  
(7)

\[ [L_{AB}, L_{CD}] = \eta_{AD} N_{BC} + \eta_{BC} N_{AD} - \eta_{AC} N_{BD} - \eta_{BD} N_{AC}, \]  
(8)

\[ [N_{ab}, L_{CD}] = 0, \]  
(9)

where the metric tensor $\eta_{AB}$ is given by

\[ \eta_{AB} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & -1 \end{bmatrix}, \]  
(10)

and the $N_{ab}$ generators read

\[ N_{ab} = J_{ab} - \frac{c}{4a^2} Z_{ab}. \]  
(11)

The $N_{ab}$ generators form a basis for the Lorentz algebra $\mathfrak{so}(D - 1, 1)$. The $L_{AB}$ generators, on the other hand, are given by

\[ L_{AB} = \begin{bmatrix} L_{ab} & L_{a,D} \\ L_{D,a} & L_{D,D} \end{bmatrix} = \begin{bmatrix} \frac{c}{4a^2} Z_{ab} & \frac{1}{2a} P_a \\ -\frac{1}{2a} P_a & 0 \end{bmatrix}. \]  
(12)