A NEW FAMILY OF MIXED METHODS FOR THE REISSNER-MINDLIN PLATE MODEL BASED ON A SYSTEM OF FIRST-ORDER EQUATIONS

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Abstract. The mixed method for the biharmonic problem introduced in [12] is extended to the Reissner-Mindlin plate model. The Reissner-Mindlin problem is written as a system of first order equations and all the resulting variables are approximated. However, the hybrid form of the method allows one to eliminate all the variables and have a final system only involving the Lagrange multipliers that approximate the transverse displacement and rotation at the edges of the triangulation. Mixed finite element spaces for elasticity with weakly imposed symmetry are used to approximate the bending moment matrix. Optimal estimates independent of the plate thickness are proved for the transverse displacement, rotation and bending moment. A post-processing technique is provided for the displacement and rotation variables and we show numerically that they converge faster than the original approximations.

1. Introduction

In [12] we developed a new mixed finite element method for the biharmonic problem. Here we develop a similar method for the more challenging hard clamped Reissner-Mindlin plate model:

$$-\nabla\cdot(\boldsymbol{C}\underline{\boldsymbol{\epsilon}}(\boldsymbol{r}))-\lambda t^{-2}(\nabla u-\boldsymbol{r})=0 \quad \text{in } \Omega, \tag{1.1a}$$

$$-\lambda t^{-2}\nabla\cdot(\nabla u-\boldsymbol{r})=f \quad \text{in } \Omega, \tag{1.1b}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{1.1c}$$

$$\boldsymbol{r}=0 \quad \text{on } \partial\Omega. \tag{1.1d}$$

where $\Omega\subset R^2$ is a polygonal domain and $f\in L^2(\Omega)$. Here t is the thickness of the plate and λ is a positive parameter. Moreover, the tensor $\boldsymbol{C}$ is defined to be

$$\boldsymbol{C}\underline{\boldsymbol{\tau}}=\frac{E}{12(1-\nu^2)}((1-\nu)\,\underline{\boldsymbol{\tau}}+\nu\,\mathrm{tr}(\underline{\boldsymbol{\tau}})\,\underline{\boldsymbol{I}}),$$

where ν is the Poisson ratio, $E=\frac{2(1+\nu)\lambda}{\kappa}$ is the Young's modulus and κ is the shear correction factor. The variable u is the transverse displacement and $\boldsymbol{r}$ the rotation.

Mixed finite elements for (1.1) typically approximate directly u and $\boldsymbol{r}$, and the shear stress $\boldsymbol{\sigma}=\lambda t^{-2}\nabla\cdot(\boldsymbol{r}-\nabla u)$; see for instance [3, 5, 10, 19, 4, 13, 17, 18, 21, 22] and [20] for

2000 *Mathematics Subject Classification.* 65M60,65N30,35L65.

Key words and phrases. finite element, Reissner-Mindlin, biharmonic, mixed method, hybrid method.

The first author was partially supported by CONICYT-Chile through the FONDECYT Grant 11070085, and by the Dirección de Investigación of the Universidad Católica de la Santísima Concepción. The second author was partially supported by the National Science Foundation (grant DMS-0914596).

a review. Instead, our method is based on the following formulation of the above problem

$$
\begin{aligned}
\boldsymbol{q} &= \nabla u, & \underline{\boldsymbol{\rho}} &= \frac{1}{2}(\nabla \boldsymbol{r} - (\nabla \boldsymbol{r})^T) && \text{in } \Omega,\\
\mathcal{A}\underline{\boldsymbol{z}} &= \nabla \boldsymbol{r} - \underline{\boldsymbol{\rho}}, & \boldsymbol{\sigma} &= \nabla \cdot \underline{\boldsymbol{z}} && \text{in } \Omega,\\
\boldsymbol{r} - \boldsymbol{q} - \hat{t}^2 \boldsymbol{\sigma} &= 0, & \nabla \cdot \boldsymbol{\sigma} &= f && \text{in } \Omega,\\
u &= 0, & \boldsymbol{r} &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

where we define $\hat{t} := \frac{t}{\sqrt{\lambda}}$ and $\mathcal{A}$ denotes the inverse of $\boldsymbol{C}$. We use the following convention $(\nabla \boldsymbol{q})_{ij} = \partial_{x_j}(q_i)$ for $1 \leq i, j \leq d$ where q_i is the i-th component of $\boldsymbol{q}$. Moreover, $(\nabla \cdot \underline{\boldsymbol{z}})_i = \sum_{j=1}^{d} \partial_{x_j} z_{ij}$ where the z_{ij} is the ij-entry of $\underline{\boldsymbol{z}}$. Although we introduced three new variables, we later will present a hybrid form of the mixed method that will allow us to eliminate all the interior variables locally to obtain a system for the Lagrange multipliers which have domain the interfaces of the triangulation. This makes the method computationally competitive. We would like to point out that Amara et al. [1] considered a low order method where they also approximate the bending moment directly which is, however, different from our lowest order method.

A desirable property for a method to have is that the approximations have provable error bounds independent of the plate thickness t. Indeed, all the methods considered in the review paper [20] have this property. Similarly, for our method we will prove optimal error estimates for the transverse displacement, rotation and bending moment independent of t.

The key idea of our method is that we formulate (1.1) such that $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and each row of $\underline{\boldsymbol{z}}$ belongs to $\boldsymbol{H}(\mathrm{div};\Omega)$ and all the other variables will only be required to be in $L^2(\Omega)$. This will allow us to use the Raviart-Thomas spaces, and in fact this is what we did in [12]. However, for the Reissner-Mindlin problem, in contrast to the biharmonic problem, we need to deal with the symmetric gradient of $\boldsymbol{r}$. We deal with this issue by using weakly symmetric elements borrowed from elasticity (see [2, 9, 16]) and this is why we introduced the anti-symmetric gradient $\underline{\boldsymbol{\rho}}$ above. By doing this we can hybridize our method and eliminate all the interior variables and only get a formulation for the Lagrange multipliers that approximate u and $\boldsymbol{r}$ on the edges of the triangulation. Hence, the final linear system that arises from our new method has exactly $3(k+1)$ (for $k \geq 1$) times the number of interior edges as unknowns if we consider Raviart-Thomas elements of index k.

We would like to mention that the analysis of the method we present in this paper for the Reissner-Mindlin problem will have many similarities with the analysis we performed for the biharmonic problem [12]. However, there are two main differences. First, here we have to prove estimates that are independent of t whereas for the biharmonic problem this is not an issue. Second, here we have to borrow some techniques for weakly symmetric methods for elasticity because of our choices of spaces which again did not arise in [12].

In [12] for the biharmonic we were able to prove that the projection of the error of the variable u superconverges with two orders higher than the optimal estimate. This allowed us to define a local post-processing procedure that produces a new approximation to u that converges with two orders more than the original approximation to u. Such estimates are based on a duality argument and certain regularity needs to be assumed. For the biharmonic problem such regularity estimates are known, however, for the Reissner-Mindlin problem the elliptic regularity results depend on the thickness t which does not