In the context of the standard special relativity, the Lorentz transformations are given by:

\[ \begin{align*}
\frac{x' \pm ct}{\gamma u} &= \frac{x \pm ct}{u} \\
\frac{y'}{\gamma u} &= \frac{y}{u} \\
\frac{z'}{\gamma u} &= \frac{z}{u}
\end{align*} \]

Where \( x' \) and \( y' \) and \( z' \) are the transformed coordinates, \( x \) and \( y \) and \( z \) are the original coordinates, and \( c \) is the speed of light. These transformations ensure that the speed of light remains constant for all observers.

The Lorentz transformations also lead to the concept of time dilation and length contraction, which are fundamental aspects of special relativity.

In the context of general relativity, the standard special relativity is modified to account for the curvature of spacetime. The Lorentz transformations are replaced by the more general coordinate transformations, which are determined by the metric of spacetime.

Abstract

In this paper, we explore the implications of the standard special relativity in the context of general relativity. We show that the standard special relativity is a good approximation in the weak field limit, but it breaks down in the strong field regime. We propose a new framework that incorporates the effects of general relativity into the standard special relativity, and we demonstrate that this framework provides a consistent description of relativistic phenomena in a variety of settings.

Keywords: Special Relativity, General Relativity, Lorentz Transformations, Spacetime Curvature, Black Holes.
If CS theories are to provide the appropriate gauge-theory framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to GR.

An interesting research in this direction has been recently carried out [18, 19]. In these references it was found that the modification of the CS theory for AdS gravity following the expansion method of Ref. [20] is not sufficient to produce a direct link with GR. In fact, it was shown that, although the action reduces to Einstein–Hilbert (EH) when the matter fields are switched off, the field equations do not. Indeed, the corresponding field equations impose severe restrictions on the geometry, which are so strong as to rule out, for instance, the five-dimensional Schwarzschild solution.

It is the purpose of this paper to show that standard, five-dimensional GR (without a cosmological constant) can be embedded in a CS theory for a certain Lie algebra $\mathfrak{g}$. The CS Lagrangian is built from a $\mathfrak{g}$-valued, one-form gauge connection $A$ [cf. eq. (26)] which depends on a scale parameter $\ell$—a coupling constant that characterizes different regimes within the theory. The $\mathfrak{g}$ algebra, on the other hand, is constructed from the AdS algebra and a particular semigroup $S$ by means of the $S$-expansion procedure introduced in Refs. [21, 22]. The field content induced by $\mathfrak{g}$ includes the vielbein $e^a$, the spin connection $\omega^{ab}$ and two extra bosonic fields $h^a$ and $k^{ab}$. The full CS field equations impose severe restrictions on the geometry [18, 19], which at a special critical point in the space of couplings ($\ell = 0$) disappear to yield pure GR.

The paper is organized as follows. In Sec. 2 we briefly review CS AdS gravity. An explicit action for five-dimensional gravity is considered in Sec. 3, where the Lie algebra $S$-expansion procedure is used to obtain a $\mathfrak{g}$-invariant CS action that includes the coupling constant $\ell$. It is then shown that the usual EH theory arises in the strict limit where the scale parameter $\ell$ equals zero. Sec. 4 concludes the work with a comment about possible developments.

2. Chern–Simons Anti-de Sitter Gravity

The CS AdS Lagrangian for gravity in $d = 2n + 1$ dimensions is given by [2, 3]

$$L_{AdS}^{(2n+1)} = \kappa e_{a_1 \cdots a_{2n+1}} \sum_{k=0}^{n} \frac{c_k}{\ell^{(n-k)+1}} R_{a_1 a_2} \cdots R^{a_{2k-1} a_{2k}} e^{a_{2k+1}} \cdots e^{a_{2n+1}}$$

where the $c_k$ constants are defined as

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k}.$$  

$e^a$ corresponds to the one-form vielbein, and $R^{ab} = d\omega^{ab} + \omega^a \omega^{cb}$ to the Riemann curvature in the first-order formalism.

The Lagrangian (5) is off-shell invariant under the AdS Lie algebra $\mathfrak{so}(2n, 2)$, whose generators $\mathbf{J}_{ab}$ of Lorentz transformations and $\mathbf{P}_a$ of AdS boosts satisfy the commutation relations

$$[\mathbf{J}_{ab}, \mathbf{J}_{cd}] = \eta_{cb} \mathbf{J}_{ad} - \eta_{ca} \mathbf{J}_{bd} + \eta_{db} \mathbf{J}_{ca} - \eta_{da} \mathbf{J}_{cb},$$

$$\mathbf{J}_{ab}, \mathbf{P}_c = \eta_{cb} \mathbf{P}_a - \eta_{ca} \mathbf{P}_b,$$

$$[\mathbf{P}_a, \mathbf{P}_b] = \mathbf{J}_{ab}.$$  

The Levi-Civita symbol $\varepsilon_{a_1 \cdots a_{2n+1}}$ in (5) is to be regarded as the only non-vanishing component of the symmetric, $\mathfrak{so}(2n, 2)$-invariant tensor of rank $r = n+1$, namely

$$\langle \mathbf{J}_{a_1 a_2} \cdots \mathbf{J}_{a_{2n-1} a_{2n}} \mathbf{P}_{a_{2n+1}} \rangle = \frac{2^n}{n+1} \varepsilon_{a_1 \cdots a_{2n+1}}.$$  

In order to interpret the gauge field associated with a translational generator $\mathbf{P}_a$ as the vielbein, one is forced to introduce a length scale $\ell$ in the theory. To see why this happens, consider the following argument. Given that (i) the exterior derivative operator $d = dx^\mu \partial_\mu$ is dimensionless, and (ii) one can always choose Lie algebra generators $T_A$ to be dimensionless as well, the one-form connection fields $A^A = A^A_\mu dx^\mu$ must also be dimensionless. However, the vielbein $e^a = e^a_\mu dx^\mu$ must have dimensions of length if it is to be related to the spacetime metric $g_{\mu \nu}$ through the usual equation $g_{\mu \nu} = e^a_\mu e^b_\nu \eta_{ab}$. This means that the “true” gauge field must be of the form $e^a / \ell$, where $\ell$ is a length.

Therefore, following Refs. [2, 3], the one-form gauge field $A$ of the CS theory is given in this case by

$$A = \frac{1}{\ell} e^a \mathbf{P}_a + \frac{1}{2} \omega^{ab} \mathbf{J}_{ab}.  \tag{11}$$

It is important to notice that once the length scale $\ell$ is brought in to the CS theory, the Lagrangian splits into several sectors, each one of them proportional to a different power of $\ell$, as we can see directly in eq. (5).

CS gravity is a well-defined gauge theory, but the presence of higher powers of the curvature makes its dynamics very remote from that for standard EH gravity. In fact, it seems very difficult to recover EH dynamics from a pure gauge, off-shell invariant theory in odd\(^3\) dimensions (see, e.g., Refs. [18, 19]).

3. Einstein–Hilbert Action from Five-dimensional Chern–Simons Gravity

In this section we show how to recover five-dimensional GR from CS Gravity. The generalization to an arbitrary odd dimension is given in Appendix A.

\(^3\)In even dimensions, the problem has been solved in a very elegant way using topological defects [23].