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A complete characterization of strong duality in nonconvex optimization with a single constraint

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Abstract We first establish sufficient conditions ensuring strong duality for cone constrained nonconvex optimization problems under a generalized Slater-type condition. Such conditions allow us to cover situations where recent results cannot be applied. Afterwards, we provide a new complete characterization of strong duality for a problem with a single constraint: showing, in particular, that strong duality still holds without the standard Slater condition. This yields Lagrange multipliers characterizations of global optimality in case of (not necessarily convex) quadratic homogeneous functions after applying a generalized joint-range convexity result. Furthermore, a result which reduces a constrained minimization problem into one with a single constraint under generalized convexity assumptions, is also presented.

Keywords Strong duality · Nonconvex optimization

Mathematics Subject Classification (2000) Primary: 90C30 · 41A65 · 52A07

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1 Introduction and formulation of the problem

Let X be a real locally convex topological vector space; Y be a normed space; $P \subseteq Y$ be a closed convex cone with possibly empty interior, and C be a subset of X. Given $f : C \to \mathbb{R}$ and $g : C \to Y$, let us consider the cone constrained minimization problem

$$\mu \doteq \inf_{\substack{g(x)\in -P \\ x\in C}} f(x). \tag{P}$$

Thus, the constraint set may be described by inequality and equality constraints. The Lagrangian dual problem associated to (P) is

$$\nu \doteq \sup_{\lambda^*\in P^*} \inf_{x\in C} [f(x) + \langle \lambda^*, g(x)\rangle], \tag{D}$$

where P^* is the non negative polar cone of P. We say Problem (P) has a (Lagrangian) *zero duality gap* if the optimal values of (P) and (D) coincide, that is, $\mu = \nu$. The Problem (P) is said to have *strong duality* if it has a zero duality gap and Problem (D) admits a solution. To characterize this property is one of the most important problems in optimization, and certainly the lack of convexity makes the task an interesting challenge in mathematics. To that purpose, some constraints qualification (CQ) are needed, which may be of Slater-type, or interior-point condition, and in some other situation it requires a closed-cone CQ. Such CQ often restrict some applications.

More precisely, when $X = \mathbb{R}^n$ and $P = [0, +\infty[$ with g being a quadratic function that is not identically zero, the authors in [15] prove that, (P) has strong duality for each quadratic function f if, and only if there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$.

Similarly, when g is P-convex (see 3.8) and continuous, it is proven in [4] that (P) has strong duality for each $f \in X^*$ if, and only if a certain CQ holds. This CQ involves the epigraph of the support function of C and the epigraph of the conjugate of the function $x \mapsto \langle \lambda^*, g(x)\rangle$. This CQ is also equivalent to (P) has strong duality for each continuous and convex function f [14]. Stable zero duality gaps in convex programming (g is continuous, P-convex, and f is lower semicontinuous proper convex function), that is, strong duality for each linear perturbation of f, were characterized in terms of a similar CQ as above, see [16,18] for details.

Apart from these characterizations several sufficient conditions of the zero duality gap have been established in the literature, see [1,2,4,6,7,12,27].

Our goal in this paper is, firstly, to derive conditions for (P) to have strong duality under no convexity assumptions. Unlike some of the above results, which involve conditions on g and C that guarantee (P) has strong duality for every f in a certain class of functions, our approach allows us to derive conditions on the pair, f and g jointly, that ensure (P) has strong duality, under no convexity assumptions; this result can be used to situations where none of the results in [4–7,12,14,16], for instance, is applicable. Secondly, we provide a new characterization of strong duality in case we have a single constraint.

By assuming that x_0 is a solution to problem (P), the authors in [6, Corollary 3.1] prove that strong duality holds if and only if condition

$$T(\tilde{M}; (f(x_0), 0)) \cap (]-\infty, 0[\times\{0\}) = \emptyset \tag{S}$$

is satisfied, where $T(A; x)$ stands for the contingent cone to A at $x \in A$, and