A-priori and a-posteriori error analysis of a wavelet-based stabilization for the mixed finite element method*

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Abstract

We use Galerkin least-squares terms and biorthogonal wavelet bases to develop a new stabilized dual-mixed finite element method for second order elliptic equations in divergence form with Neumann boundary conditions. The approach introduces the trace of the solution on the boundary as a new unknown that acts also as a Lagrange multiplier. We show that the resulting stabilized dual-mixed variational formulation and the associated discrete scheme defined with Raviart-Thomas spaces are well posed, and derive the usual a-priori error estimates and the corresponding rate of convergence. Furthermore, a reliable and efficient residual based a-posteriori error estimator and a reliable and quasi-efficient one are provided.

Key words. Mixed finite elements, biorthogonal wavelet bases, Raviart-Thomas spaces, a-posteriori error estimators.

1 Introduction

In the recent paper [2], the a-priori and a-posteriori error analysis of an augmented mixed finite element method with Lagrange multipliers, as applied to elliptic equations in divergence form with mixed boundary conditions, is introduced and analyzed. Following [1], the approach in [2] imposes first the essential (Neumann) boundary condition in a weak sense, which motivates the introduction of a further Lagrange multiplier given precisely by the trace of the solution on the Neumann boundary. Then, the augmented scheme is obtained by including Galerkin-least squares terms arising from the constitutive and equilibrium equations, which, however, still yields a saddle point operator equation as the resulting variational formulation. Hence, the classical Babuška-Brezzi theory is applied to show that the continuous formulation and its associated Galerkin scheme defined with Raviart-Thomas spaces are well posed. The corresponding a-priori error estimates and rates of convergence are also derived. In addition, a reliable and efficient residual based a-posteriori error estimator and a reliable and quasi-efficient Ritz projection based

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*This research was partially supported by CONICYT-Chile through the FONDAP Program in Applied Mathematics, and by the Dirección de Investigación of the Universidad de Concepción through the Advanced Research Groups Program

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one are provided in [2]. It is important to emphasize that the application of the Babuška-Brezzi theory to saddle point variational formulations usually restricts the possible choices of the finite element subspaces and imposes certain assumptions on the corresponding mesh sizes. This is particularly necessary for the verification of the discrete inf-sup conditions, which are the main ingredients of the corresponding analysis.

The purpose of the present work, which is an extended and completed version of our previous paper [3], is to modify the method from [2] so that the resulting variational formulation becomes a bounded and elliptic operator equation. In this way, it just suffices to apply the well known Lax-Milgram theorem to prove the well-posedness of the continuous and discrete schemes, and hence no additional conditions on the subspaces or the meshesizes are required. In order to achieve the above, we propose here to further augment the dual-mixed formulation from [2] by incorporating the residual associated to the boundary trace. More precisely, the new feature of our approach lies on the fact that this boundary residual is measured in terms of the equivalent Sobolev norm of order $1/2$ introduced in [4], which is computed by means of biorthogonal wavelet bases. In addition, we also develop here an a-posteriori error analysis yielding a reliable and efficient residual based estimator and a reliable and quasi-efficient one. The outline of the paper is as follows. In Section 2 we introduce the stabilized dual-mixed variational formulation and establish its unique solvability and stability. Next, in Section 3 we introduce the stabilized mixed finite element scheme and prove that it is also well-posed. We remark here that this scheme makes use of a discrete version of the above mentioned equivalent Sobolev norm, which arises after a suitable finite truncation of the wavelet bases. Finally, in Section 4 we provide the a-posteriori error analysis of our augmented scheme.

Throughout the rest of the paper we utilize the standard terminology for Sobolev spaces, norms, and seminorms, and use $C$ and $c$, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The stabilized dual-mixed variational formulation

We first describe the model boundary value problem and introduce the corresponding dual-mixed variational formulation. Let $\Omega$ be a simply connected and bounded domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma$. Then, given $f \in L^2(\Omega)$, $g \in H^{-1/2}(\Gamma)$, and a matrix valued function $\kappa \in C(\overline{\Omega})$, we seek $u \in H^1(\Omega)$ such that

$$-\text{div}(\kappa \nabla u) = f \quad \text{in} \quad \Omega, \quad \kappa \nabla u \cdot \nu = g \quad \text{on} \quad \Gamma, \quad (2.1)$$

where $\nu$ is the unit outward normal vector to $\Gamma$. We assume that $\kappa$ is symmetric and uniformly positive definite, that is, there exists $\alpha > 0$ such that

$$(\kappa(x) z) \cdot z \geq \alpha \|z\|^2 \quad \forall x \in \Omega \quad \forall z \in \mathbb{R}^2, \quad (2.2)$$

which implies the following inequalities

$$(\kappa^{-1}(x) z) \cdot z \geq \kappa^{-1}(x) \|z\| \quad \text{and} \quad \|\kappa^{-1}(x) z\| \leq \frac{1}{\alpha} \|z\| \quad \forall x \in \Omega, \quad \forall z \in \mathbb{R}^2. \quad (2.3)$$

Moreover, since $\kappa \in C(\overline{\Omega})$ we deduce the existence of $M > 0$ such that

$$\|\kappa(x) z\| \leq M \|z\| \quad \text{and} \quad \frac{1}{M} \|z\| \leq \|\kappa^{-1}(x) z\| \quad \forall x \in \Omega, \quad \forall z \in \mathbb{R}^2, \quad (2.4)$$

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