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An adaptive stabilized finite element scheme for a water quality model

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Abstract

Residual type *a posteriori* error estimators are introduced in this paper for an advection–diffusion–reaction problem with a Dirac delta source term. The error is measured in an adequately weighted $W^{1,p}$-norm. These estimators are proved to yield global upper and local lower bounds for the corresponding norms of the error. They are used to guide adaptive procedures, which are experimentally shown to lead to optimal orders of convergence.

Keywords: Advection–reaction–diffusion problem; Dirac delta source term; *A posteriori* error estimates; Stabilized finite elements

1. Introduction

This paper deals with the advection–diffusion–reaction equation with a Dirac delta source term. This kind of problems arise, for example, in modeling pollutant transport and degradation in an aquatic media if the pollution source is a single point. In particular, our work is motivated by the need of an efficient scheme to be used in a water quality model for the river Bío-Bío in Chile.

It is simple to show that the solution of this problem belongs to L^p for $1 \leqslant p < \infty$ and to $W^{1,p}$ for $p < 2$. In spite of the fact that the solution does not belong to H^1, this problem can be numerically approximated by standard finite elements.

Specially interesting is the case when the advective term is dominant, as typically happens in real problems. In this case, the solution of the equation has a strong interior layer arising from the source point aligned with the velocity direction. The standard Galerkin approximation usually fails in this situation because this method introduces non-physical oscillations.

A possible remedy for this situation is to add to the variational formulation some numerical diffusion terms to stabilize the finite element solution. Some examples of this approach are the streamline upwind Petrov–Galerkin method (SUPG) (see [6]), the Galerkin least squares approximation (GLS) (see [10]), the Douglas–Wang method (see [8]), the unusual stabilized finite element method (USFEM) (see [11]) and the residual-free bubbles approximation (RFB) (see [5]). The drawback with most of these methods is that the amount of numerical diffusion added to the discretization tends to be large. This means that the solution layers are not always very well resolved because the layer zone is artificially wide. Furthermore, all this stabilization techniques do not consider non-regular right-hand sides as, for example, a Dirac delta measure.

Due to the nature of the solution, when a strong interior layer is present, it is convenient to compute the numerical solution in a well adapted mesh, which should be obtained by means of an adaptive scheme.

There are not many references in the literature dealing with *a posteriori* techniques for this equation. The reason of this is that most of the standard error estimators involve equivalence constants depending on negative powers of the

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diffusion parameter, which leads to very poor results in the advection or reaction dominated cases. An error estimator which is robust in the sense of leading to global upper and local lower bounds depending at most on the local mesh Peclet number has been developed by Verfürth (see [17,18]). Using these results, Sangalli has analyzed a residual *a posteriori* error estimate for the residual-free bubbles scheme (see [15]). On the other hand, Knop et al. have developed some *a posteriori* error estimates using a stabilized scheme combined with a shock capture technique to control the local oscillations in the crosswind direction (see [13]). Finally, Wang has introduced an error estimate for the advection–diffusion equation based on the solution of local problems on each element of the triangulation (see [19]). In all these works smooth source terms are considered. On the other hand, an *a posteriori* error analysis has been recently developed in [3] for the Laplace equation with a delta source term. To the best of the authors knowledge, no *a posteriori* error analysis has been performed for the advection–diffusion–reaction equation with a non-regular right-hand side.

In this paper, we introduce and analyze from theoretical and experimental points of view an adaptive scheme to efficiently solve the advection–reaction–diffusion equation with a Dirac delta source term. This scheme is based on the stabilized finite element method introduced in [11], combined with an error estimator similar to that developed in [2,17]. Although the stabilization technique [11] has been analyzed only for regular right-hand sides, our experiments show that the numerical scheme is convergent also in our case. Under appropriate assumptions, we prove global upper and local lower error estimates in a weighted $W^{1,p}$-norm, with constants which depend on the shape-regularity of the mesh, the polynomial degree of the finite element approximating space, and, eventually, on the diffusion parameter. Because of this last dependence, our theoretical results are not optimal. However, we perform several numerical experiments in order to show the effectiveness of our approach to capture the layers very sharply and without significant oscillations.

The paper is organized as follows. In Section 2 we recall the advection–diffusion–reaction problem under consideration and the stabilized scheme. In Section 3 we define an *a posteriori* error estimator, prove some technical lemmas and show its equivalence with the norm of the finite element approximation error. Finally, in Section 4, we introduce the adaptive scheme and report the results of some numerical tests which allow us to asses the performance of our approach.

2. A stabilized method for a model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a Lipschitz boundary $\Gamma = \overline{\Gamma}_{\rm D} \cup \overline{\Gamma}_{\rm N}$, with $\Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset$. We denote by $\boldsymbol{n}$ the outer unit normal vector to Γ. Let δ_{x_0} be the Dirac delta measure supported at an inner point $x_0 \in \Omega$.

Our model problem is the advection–reaction–diffusion equation

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u + bu = \delta_{x_0} & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\rm D}, \\ \varepsilon\frac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_{\rm N}, \end{cases} \tag{2.1}$$

where:

(A1) $\varepsilon \in \mathbb{R} : \varepsilon > 0$;
(A2) $\boldsymbol{a} \in W^{1,\infty}(\Omega)^2$, $-\frac{1}{2}\mathrm{div}\boldsymbol{a} + b \geqslant 0$;
(A3) $b \in \mathbb{R}$, $b \geqslant 0$;
(A4) $\Gamma_{\rm D} \supset \{\boldsymbol{x} \in \Gamma : \boldsymbol{a}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$;
(A5) $g \in L^2(\Gamma_{\rm N})$;
(A6) either $b > 0$ or $|\Gamma_{\rm D}| > 0$.

We are interested in the advection–reaction dominated case in which $\varepsilon \ll \|\boldsymbol{a}\|_{0,\infty,\Omega} + b$.

Here and thereafter we use standard notation for Sobolev and Lebesgue spaces and norms. Moreover, let $W_D^{1,r}(\Omega) := \{\varphi \in W^{1,r}(\Omega) : \varphi|_{\Gamma_{\rm D}} = 0\}$, $1 < r < \infty$.

Let us remark that problem (2.1) does not have a solution in $H^1(\Omega)$. However, it has a solution in $W^{1,p}(\Omega)$ $\forall p < 2$. In fact, $G(x) := \frac{1}{2\pi}\log|x - x_0|$ is such that

$$-\Delta G = \delta_{x_0} \quad \text{in } \Omega;$$

i.e., $-G$ is a fundamental solution of the Laplace operator. Straightforward calculations show that $G \in W^{1,p}(\Omega)$ $\forall p \in [1,2)$. Hence, substituting $u = w + \varepsilon^{-1}G$ in (2.1), we observe that problem (2.1) has a unique solution if and only if the following problem does:

$$\begin{cases} -\varepsilon\Delta w + \boldsymbol{a}\cdot\nabla w + bw = -\varepsilon^{-1}\boldsymbol{a}\cdot\nabla G - \varepsilon^{-1}bG & \text{in } \Omega, \\ w = -\varepsilon^{-1}G & \text{on } \Gamma_{\rm D}, \\ \varepsilon\frac{\partial w}{\partial \boldsymbol{n}} = g - \frac{\partial G}{\partial \boldsymbol{n}} & \text{on } \Gamma_{\rm N}. \end{cases} \tag{2.2}$$

Since $x_0 \notin \partial\Omega$, G and its normal derivative are smooth on $\partial\Omega$. Hence, standard arguments show that problem (2.2) has a unique solution $w \in H^1(\Omega)$ (see for instance [14]). Moreover, according to the results of [12], problem (2.2) has no other solution in $W^{1,p}(\Omega)$ for $p \in (p^*, 2)$, where $p^* := \frac{2}{1+\pi/(2\omega)}$, with ω being the largest reentrant corner of the domain Ω.

Consequently, problem (2.1) has a solution $u \in W^{1,p}(\Omega)$ $\forall p < 2$, and this solution is unique if $p^* \leqslant p < 2$. Let us remark that for any polygonal domain Ω, $p^* \leqslant 8/5$. Moreover, if Ω is convex, then $p^* < 4/3$. From now on we restrict our analysis to a fixed $p \in (p^*, 2)$. Moreover, let $q \in (2, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

Let B be the bilinear form defined on $W_D^{1,p}(\Omega) \times W_D^{1,q}(\Omega)$ by

$$B(v,w) := \int_\Omega (\varepsilon\nabla v\cdot\nabla w + \boldsymbol{a}\cdot\nabla v\; w + bvw). \tag{2.3}$$