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# CONTROLS INSENSITIZING THE OBSERVATION OF A QUASI-GEOSTROPHIC OCEAN MODEL*

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**Abstract.** We consider a linear quasi-geostrophic ocean model with partially known initial conditions. We search for controls that make the observation locally insensitive to the perturbations of the initial data. Their existence is equivalent to the null controllability property for an associated cascade Stokes-like system. Thanks to the presence of the Coriolis term, we are able to prove the existence of such controls. Our strategy is the following. First, we prove a unique continuation property for the adjoint of the state system that leads to approximate controllability; then, under certain assumptions, an observability inequality is established for the adjoint. The proof is inspired by the arguments leading to the unique continuation property. This inequality leads to the desired null controllability result.

**Key words.** insensitizing controls, Carleman inequalities, unique continuation, null controllability, ocean model

**AMS subject classifications.** 93B05, 35B37, 35B60, 35Q30

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## 1. Introduction and main results.

**1.1. Incomplete initial data ocean model.** Let $\Omega$ be a nonempty open bounded and connected subset of $\mathbb{R}^2$, with boundary $\Gamma$ of class $\mathcal{C}^2$ and outwards unit normal vector $\nu = \nu(x)$. Let $\omega$ be a nonempty open subset of $\Omega$, $T > 0$, $Q = \Omega \times (0,T)$, and $\Sigma = \Gamma \times (0,T)$. In this paper, we will consider a linear quasi-geostrophic ocean model [1, 15, 16] described by the following equations:

$$
\begin{cases} u_t - A\Delta u + \gamma u + (f_0 + \beta x_2)\,\mathrm{k}\wedge u + \dfrac{1}{\rho_0}\nabla p = \mathcal{T} + h1_\omega & \text{in } Q,\\ \operatorname{div} u = 0 & \text{in } Q,\\ u = 0 & \text{on } \Sigma,\\ u(0) = u_0 + \tau\widehat{u}_0 & \text{in } \Omega, \end{cases} \tag{1.1}
$$

where $u(x,t)$ and $p(x,t)$, respectively, denote the velocity and the pressure of the fluid at $(x,t) = (x_1,x_2,t) \in \mathbb{R}^2 \times \mathbb{R}_+$. In this model, $A$ represents the horizontal *eddy viscosity* coefficient, $\gamma$ is the bottom *friction* coefficient, $\rho_0$ is the fluid density, and $(f_0 + \beta x_2)\mathrm{k}\wedge u$ is the Coriolis term, with $\mathrm{k}\wedge u = (-u_2, u_1)$. In the right-hand side, $1_\omega$ denotes the characteristic function of $\omega$ and $\mathcal{T}$ is a given source in $L^2(Q)^2$. The term

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$\tau\widehat{u}_0$, where $\tau \in \mathbb{R}$, represents a small unknown perturbation of the initial velocity field $u_0$, and $h = h(x,t)$ is a control function to be determined.

Notice that the Coriolis force is represented by a zero order coupling term in the equations. It introduces a different behavior of the system depending on the direction in space. To simplify the presentation of the results, we will assume that $A=1$, $\gamma=1$, $f_0 = 1$, $\beta = 1$, and $\rho_0 = 1$.

We introduce the following spaces, which are usual in the analysis of Stokes systems:

$$H = \{v \in L^2(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot \nu = 0 \text{ on } \Gamma\},$$
$$V = \{v \in H_0^1(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega\}, \qquad W = H^2(\Omega)^2 \cap V.$$

Recall that

$$W \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow W',$$

where the embeddings are dense and compact.

For any given $u_0, \tau\widehat{u}_0 \in H$ with $\|\widehat{u}_0\|_{0,\Omega} = 1$, any $\mathcal{T} \in L^2(Q)^2$, and any $h \in L^2(\omega \times (0,T))^2$, the linear system (1.1) possesses a unique solution $(u,p)$, with $u \in L^2(0,T;V) \cap H^1(0,T;V')$ and $p \in W^{-1,\infty}(0,T;L^2(\Omega))$. ($p$ is unique up to an additive distribution only depending on $t$.) This is easily proved by adapting the arguments of [17] to the presence of a skew-symmetric Coriolis term in the equations. Notice that if we had $u_0 + \tau\widehat{u}_0 \in V$, then the couple $(u,p)$ would satisfy $u \in L^2(0,T;W) \cap H^1(0,T;H)$ and $p \in L^2(0,T;H^1(\Omega))$.

We will be concerned with the search of controls such that the velocity measurements over an observation set are either insensitive or almost insensitive to small variations of the initial conditions. To do this, we will use *insensitizing control theory.*

**1.2. Insensitizing controls and controllability.** Let $\mathcal{O}$ be an open nonempty subset of $\Omega$ and let us introduce the following functional, defined on the family of solutions to (1.1):

$$\Phi(u) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |u(x,t)|^2 \, dx\, dt. \tag{1.2}$$

The notion of *insensitizing controls* was introduced by Lions [13]. In the context of (1.1)–(1.2), it reads as follows.

DEFINITION 1.1. *We say that the control $h \in L^2(\omega \times (0,T))^2$ is $\Phi$ insensitizing if*

$$\frac{d}{d\tau}\Phi(u)\Big|_{\tau=0} = 0 \qquad \forall \widehat{u}_0 \in H \quad \textit{with} \quad \|\widehat{u}_0\|_{0,\Omega} = 1. \tag{1.3}$$

*On the other hand, we say that $h \in L^2(\omega \times (0,T))^2$ is $\Phi$ $\varepsilon$-insensitizing if*

$$\left|\frac{d}{d\tau}\Phi(u)\Big|_{\tau=0}\right| \le \varepsilon \qquad \forall \widehat{u}_0 \in H \quad \textit{with} \quad \|\widehat{u}_0\|_{0,\Omega} = 1. \tag{1.4}$$

Of course, in (1.3) and in (1.4) $u$ is, together with $p$, the solution to (1.1).

The $\Phi$ insensitizing (resp., $\Phi$ $\varepsilon$-insensitizing) controls $h$ must be interpreted as those leading to an observation $\Phi(u)$ that is locally independent (resp., almost independent) at the initial perturbation $\tau\widehat{u}_0$. The existence of such controls is a pertinent