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On the *a priori* and *a posteriori* error analysis of a two-fold saddle-point approach for nonlinear incompressible elasticity

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SUMMARY

In this paper, we reconsider the *a priori* and *a posteriori* error analysis of a new mixed finite element method for nonlinear incompressible elasticity with mixed boundary conditions. The approach, being based only on the fact that the resulting variational formulation becomes a two-fold saddle-point operator equation, simplifies the analysis and improves the results provided recently in a previous work. Thus, a well-known generalization of the classical Babuška–Brezzi theory is applied to show the well-posedness of the continuous and discrete formulations, and to derive the corresponding *a priori* error estimate. In particular, enriched PEERS subspaces are required for the solvability and stability of the associated Galerkin scheme. In addition, we use the Ritz projection operator to obtain a new reliable and quasi-efficient *a posteriori* error estimate. Finally, several numerical results illustrating the good performance of the associated adaptive algorithm are presented. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In Reference [1], we introduced and analysed a dual-mixed finite element method for nonlinear incompressible elasticity with Dirichlet boundary conditions. Our approach there

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follows previous works on dual-mixed methods for nonlinear boundary value problems (see, e.g. References [2–5]) and defines the strain tensor and the rotation as additional unknowns. The corresponding variational formulation becomes a two-fold saddle-point operator equation, and hence the generalization of the Babuška–Brezzi theory developed in Reference [6] is applied to prove that the continuous and discrete schemes are well posed. We emphasize that in this dual-mixed setting the Dirichlet boundary condition becomes natural, and therefore the equation $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0$, where $\mathrm{tr}(\boldsymbol{\sigma})$ denotes the trace of the stress tensor $\boldsymbol{\sigma}$, is introduced to guarantee the unique solvability of the formulations (see also References [7–9] where this constraint has been utilized before). A suitable enrichment of the well-known PEERS finite element subspace is then employed to guarantee that the associated Galerkin scheme is well posed. We also develop in Reference [1] a local problems-based *a posteriori* error analysis, which combines the classical Bank–Weiser approach from Reference [10] with the technique from References [11–13].

On the other hand, in the recent thesis [14] the results from Reference [1] were extended to the case of mixed boundary conditions, in which no additional uniqueness constraint is needed for well posedness. However, differently from Reference [1], the analysis in Reference [14] was performed through an equivalent three-fold saddle-point operator equation, whence a further generalization of the theory from Reference [6] was required there. In addition, a local problems-based *a posteriori* error analysis, following the same lines of Reference [1], is also developed in Reference [14].

In the present work, we simplify the presentation from Reference [14] and improve the results provided there. More precisely, we show that the analysis can be carried out by simply keeping the two-fold saddle-point structure of the original variational formulation and applying directly the abstract results from Reference [6]. In addition, a suitable upper bound of the Ritz projection is employed now to derive a new *a posteriori* error estimate, which is also shown to be reliable and quasi-efficient. The rest of the paper is organized as follows. In Section 2, we introduce the model problem, show that the associated dual-mixed variational formulation can be written as a two-fold saddle-point operator equation, collect the main results of the generalized Babuška–Brezzi theory developed in Reference [6] (see also References [15, 16]) and apply this theory to prove the unique solvability of our continuous formulation. Then, in Section 3, we define the associated Galerkin scheme, apply again the abstract theory from Reference [6] to show that it is well posed, and derive the corresponding *a priori* error estimate. Next, Section 4, deals with a Ritz projection based *a posteriori* error analysis of the formulation. Finally, several numerical results illustrating the good performance of the resulting adaptive algorithm are reported in Section 5.

We end Section 1 by specifying some notations and definitions to be employed throughout the remaining sections. Given any Hilbert space H, we denote by H^2 and $H^{2\times 2}$ the spaces of vectors and tensors of order two, respectively, with entries in H, provided with the product norms induced by the norm of H. In addition, for any $\boldsymbol{\tau} := (\tau_{ij}), \boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we denote $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}$. The deviator of tensor $\boldsymbol{\tau}$ is denoted by $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$, which satisfies $\mathrm{tr}(\boldsymbol{\tau}^{\mathrm{d}}) = 0$. Also, the superscript '$\mathtt{t}$' is used to refer to the transpose of vectors and tensors. On the other hand, we recall that $H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}\,\boldsymbol{\tau} \in [L^2(\Omega)]^2\}$ is a Hilbert space with the inner product $\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} := \langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{[L^2(\Omega)]^{2\times 2}} + \langle \mathbf{div}\,\boldsymbol{\zeta}, \mathbf{div}\,\boldsymbol{\tau}\rangle_{[L^2(\Omega)]^2}$, where $\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{[L^2(\Omega)]^{2\times 2}} := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}$ for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$, and $\langle \mathbf{v}, \mathbf{w}\rangle_{[L^2(\Omega)]^2} := \int_\Omega \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in [L^2(\Omega)]^2$. The corresponding induced norms are denoted by $\|\cdot\|_{H(\mathbf{div};\Omega)}$, $\|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$, and $\|\cdot\|_{[L^2(\Omega)]^2}$, respectively. Finally, c and C, with or without subscripts, bars, tildes or hats, denote positive constants, independent of the parameters and functions involved, which may take different values at different occurrences.

2. THE DUAL-MIXED VARIATIONAL FORMULATION

2.1. The model problem and the operator equation

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ, and such that all its interior angles lie in $]0, 2\pi[$. Also, let Γ_D and Γ_N be disjoint open subsets of Γ, with $|\Gamma_D|, |\Gamma_N| \neq 0$, such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. The model problem consists in determining the displacement $\mathbf{u} := (u_1, u_2)^{\mathrm{t}}$ and the pressure-like unknown p of an incompressible material occupying the region Ω, under the action of some external forces. More precisely, if $\boldsymbol{\sigma}(\mathbf{u}, p)$, $\mathbf{e}(\mathbf{u})$, and $\mathbf{I} \in \mathbb{R}^{2\times 2}$ denote the Cauchy tensor, the strain tensor of small deformations, and the identity matrix, respectively, the constitutive equation is given by

$$\boldsymbol{\sigma}(\mathbf{u}, p) = \mathscr{N}(\mathbf{e}(\mathbf{u})) + p\,\mathbf{I} \quad \text{in } \Omega$$

where $\mathscr{N} : [L^2(\Omega)]^{2\times 2} \to [L^2(\Omega)]^{2\times 2}$ is a nonlinear operator such that $\mathscr{N}(\mathbf{s}) = \mathscr{N}(\mathbf{s})^{\mathrm{t}}$ for each symmetric tensor $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$. We recall here that $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{t}})$. In addition, we assume that $\mathscr{N}$ induces a strongly monotone and Lipschitz continuous operator from $[L^2(\Omega)]^{2\times 2}$ into its dual, which means that there exist α_1, $\alpha_2 > 0$ such that for all $\mathbf{r}$, $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}$ there hold

$$\int_\Omega [\mathscr{N}(\mathbf{r}) - \mathscr{N}(\mathbf{s})] : [\mathbf{r} - \mathbf{s}] \geqslant \alpha_1 \|\mathbf{r} - \mathbf{s}\|^2_{[L^2(\Omega)]^{2\times 2}} \tag{1}$$

and

$$\|\mathscr{N}(\mathbf{r}) - \mathscr{N}(\mathbf{s})\|_{[L^2(\Omega)]^{2\times 2}} \leqslant \alpha_2 \|\mathbf{r} - \mathbf{s}\|_{[L^2(\Omega)]^{2\times 2}} \tag{2}$$

Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{-1/2}(\Gamma_N)]^2$, our nonlinear boundary value problem reads as follows: Find $(\boldsymbol{\sigma}, \mathbf{u}, p)$ in appropriate spaces such that

$$\begin{aligned} &\boldsymbol{\sigma} = \mathscr{N}(\mathbf{e}(\mathbf{u})) + p\mathbf{I} \quad \text{in } \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in } \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \\ &\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{g} \quad \text{on } \Gamma_N \end{aligned} \tag{3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to Γ_N and $\mathbf{div}$ denotes the usual divergence operator div acting along each row of the corresponding tensor. We recall here that the Sobolev space $[H^{-1/2}(\Gamma_N)]^2$ is the dual of $[H^{1/2}_{00}(\Gamma_N)]^2 := \{\mathbf{v}|_{\Gamma_N} : \mathbf{v} \in [H^1(\Omega)]^2, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$, and denote by $\langle\cdot,\cdot\rangle_{\Gamma_N}$ the corresponding duality pairing with respect to the $[L^2(\Gamma_N)]^2$-inner product.

We now introduce the further unknowns $\mathbf{t} := \mathbf{e}(\mathbf{u})$ in Ω, $\boldsymbol{\gamma} := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}})$ in Ω, and $\boldsymbol{\xi} := -\mathbf{u}$ on Γ_N. The tensor $\boldsymbol{\gamma}$ represents rotations and lives in the space $\mathscr{R} := \{\boldsymbol{\delta} \in [L^2(\Omega)]^{2\times 2} : \boldsymbol{\delta} + \boldsymbol{\delta}^{\mathrm{t}} = \mathbf{0}\}$, which is equipped with the norm $\|\cdot\|_{\mathscr{R}} := \|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$. In addition, we remark that $\boldsymbol{\gamma}$ and $\boldsymbol{\xi}$ play the role of the Lagrange multipliers associated to the symmetry of $\boldsymbol{\sigma}$ and the Neumann condition, respectively. Then, proceeding as in Reference [14] (see also Reference [1]), we arrive to the following variational formulation of the boundary value problem (3): Find

$(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in \mathscr{X}_1 \times \mathscr{M}_1 \times \mathscr{M}$ such that

$$\begin{aligned} \int_\Omega \mathscr{N}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \int_\Omega p \operatorname{tr}(\mathbf{s}) &= 0 \\ -\int_\Omega \mathbf{t} : \boldsymbol{\tau} + \int_\Omega q \operatorname{tr}(\mathbf{t}) - \int_\Omega \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\,\mathbf{v}, \boldsymbol{\xi} \rangle_{\Gamma_N} &= 0 \\ -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma} - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma} - \langle \boldsymbol{\sigma}\mathbf{v}, \boldsymbol{\lambda} \rangle_{\Gamma_N} &= \mathscr{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \end{aligned} \tag{4}$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})) \in \mathscr{X}_1 \times \mathscr{M}_1 \times \mathscr{M}$, where $\mathscr{X}_1 := [L^2(\Omega)]^{2\times 2}$, $\mathscr{M}_1 := H(\mathbf{div}; \Omega)\, L^2(\Omega)$, $\mathscr{M} := [L^2(\Omega)]^2 \times \mathscr{R} \times [H^{1/2}_{00}(\Gamma_N)]^2$, and $\mathscr{G} \in \mathscr{M}'$ is defined by

$$\mathscr{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) := \int_\Omega \mathbf{f} \cdot \mathbf{v} - \langle \mathbf{g}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \quad \forall (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \mathscr{M}$$

In particular, we observe that the second equation of (4) follows using that $\mathbf{t} = \nabla \mathbf{u} - \boldsymbol{\gamma}$ in Ω, $\operatorname{tr}(\mathbf{t}) = \operatorname{div} \mathbf{u} = 0$ in Ω, and $\mathbf{u} = \mathbf{0}$ on Γ_D.

Next, we show that (4) can be rewritten as a two-fold saddle-point operator equation. To this end, let $\mathscr{O}$ denote a generic null operator/functional, and define the operators $\mathscr{A}_1 : \mathscr{X}_1 \to \mathscr{X}_1'$, $\mathscr{B}_1 : \mathscr{X}_1 \to \mathscr{M}_1'$, and $\mathscr{B} : \mathscr{M}_1 \to \mathscr{M}'$, as follows:

$$\begin{aligned} [\mathscr{A}_1(\mathbf{r}), \mathbf{s}] &:= \int_\Omega \mathscr{N}(\mathbf{r}) : \mathbf{s} && \forall \mathbf{r}, \mathbf{s} \in \mathscr{X}_1 \\ [\mathscr{B}_1(\mathbf{r}), (\boldsymbol{\tau}, q)] &:= -\int_\Omega \mathbf{r} : \boldsymbol{\tau} + \int_\Omega q \operatorname{tr}(\mathbf{r}) && \forall \mathbf{r} \in \mathscr{X}_1 \ \ \forall (\boldsymbol{\tau}, q) \in \mathscr{M}_1 \end{aligned} \tag{5}$$

and

$$[\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\mathbf{v}, \boldsymbol{\lambda} \rangle_{\Gamma_N} \quad \forall (\boldsymbol{\tau}, q) \in \mathscr{M}_1 \ \ \forall (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \mathscr{M}$$

Hereafter, $[\cdot, \cdot]$ stands for the duality pairing induced by the operators involved. Also, we notice that $\mathscr{A}_1$ is nonlinear and that $\mathscr{B}_1$, $\mathscr{B}$, and the corresponding transposes $\mathscr{B}_1' : \mathscr{M}_1 \to \mathscr{X}_1'$ and $\mathscr{B}' : \mathscr{M} \to \mathscr{M}_1'$, are all linear and bounded operators.

According to the above notations, the mixed variational formulation (4) can be stated as the two-fold saddle-point operator equation: Find $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) \in \mathscr{X}_1 \times \mathscr{M}_1 \times \mathscr{M}$ such that

$$\begin{bmatrix} \mathscr{A}_1 & \mathscr{B}_1' & \mathscr{O} \\ \mathscr{B}_1 & \mathscr{O} & \mathscr{B}' \\ \mathscr{O} & \mathscr{B} & \mathscr{O} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ (\boldsymbol{\sigma}, p) \\ (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi}) \end{bmatrix} = \begin{bmatrix} \mathscr{O} \\ \mathscr{O} \\ \mathscr{G} \end{bmatrix} \tag{6}$$

The abstract theory for this kind of continuous formulation is already available (see, e.g. References [6, 15]), and its main results are collected in the following subsection.

2.2. Abstract theory for two-fold saddle-point operator equations

Let X_1, M_1, and M be Hilbert spaces, and consider a nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$, and linear bounded operators $\mathbf{B}_1 : X_1 \to M_1'$ and $\mathbf{B} : M_1 \to M'$, with transposes $\mathbf{B}_1' : M_1 \to X_1'$ and $\mathbf{B}' : M \to M_1'$, respectively. Then, we are interested in the following nonlinear variational problem: *Given* $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$, *find* $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ *such that*

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1' & \mathbf{O} \\ \mathbf{B}_1 & \mathbf{O} & \mathbf{B}' \\ \mathbf{O} & \mathbf{B} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \boldsymbol{\sigma} \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{G} \\ \mathbf{F} \end{bmatrix} \tag{7}$$

We have the following theorem.

Theorem 2.1

Let $\tilde{M}_1 := \ker(\mathbf{B})$, define $V_1 := \{\mathbf{s} \in X_1 : [\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}] = 0 \ \forall \boldsymbol{\tau} \in \tilde{M}_1\}$, and let $\Pi_1 : X_1' \to V_1'$ be the canonical imbedding defined by $\Pi_1(\mathbf{H}) = \mathbf{H}|_{V_1}$ for all $\mathbf{H} \in X_1'$. Assume that

(i) the nonlinear operator $\mathbf{A}_1 : X_1 \to X_1'$ is Lipschitz continuous with a Lipschitz constant $\gamma>0$, and for any $\tilde{\mathbf{t}} \in X_1$, the nonlinear operator $\Pi_1 \mathbf{A}_1(\cdot + \tilde{\mathbf{t}}) : V_1 \to V_1'$ is strongly monotone with a monotonicity constant $\alpha>0$ independent of $\tilde{\mathbf{t}}$.

(ii) There exists $\beta>0$ such that for all $v \in M$

$$\sup_{\substack{\boldsymbol{\tau} \in M_1 \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{\|\boldsymbol{\tau}\|_{M_1}} \geqslant \beta \, \|v\|_M \tag{8}$$

(iii) there exists $\beta_1>0$ such that for all $\boldsymbol{\tau} \in \tilde{M}_1$

$$\sup_{\substack{\mathbf{s} \in X_1 \\ \mathbf{s} \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{X_1}} \geqslant \beta_1 \, \|\boldsymbol{\tau}\|_{M_1} \tag{9}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ there exists a unique $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ solution of (7). Moreover, there exists $C>0$, independent of the solution, such that

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u)\|_{X_1 \times M_1 \times M} \leqslant C\{\|\mathbf{H}\| + \|\mathbf{G}\| + \|\mathbf{F}\| + \|\mathbf{A}_1(0)\|\}$$

Proof

See Theorem 2.4 in Reference [6] (see also References [15, Theorem 1], [16, Theorem 2.1], or [5, Theorem 4.1]). □

Now, let $X_{1,h}$, $M_{1,h}$ and M_h be finite-dimensional subspaces of X_1, M_1 and M, respectively. Then the Galerkin scheme associated with (7) reads as follows: *Given*

$(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$, *find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ *such that*

$$\begin{aligned} [\mathbf{A}_1(\mathbf{t}_h), \mathbf{s}_h] + [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &= [\mathbf{H}, \mathbf{s}_h] \\ [\mathbf{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\tau}_h), u_h] &= [\mathbf{G}, \boldsymbol{\tau}_h] \\ [\mathbf{B}(\boldsymbol{\sigma}_h), v_h] &= [\mathbf{F}, v_h] \end{aligned} \tag{10}$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \in X_{1,h} \times M_{1,h} \times M_h$.

The discrete analogue of Theorem 2.1 is established next.

Theorem 2.2

Let $\tilde{M}_{1,h} := \{\boldsymbol{\tau}_h \in M_{1,h} : [\mathbf{B}(\boldsymbol{\tau}_h), v_h] = 0 \ \ \forall v_h \in M_h\}$, define $V_{1,h} := \{\mathbf{s}_h \in X_{1,h} : [\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h] = 0 \ \ \forall \boldsymbol{\tau}_h \in \tilde{M}_{1,h}\}$ and let $\Pi_{1,h} : X_{1,h}' \to V_{1,h}'$ be the canonical imbedding. In addition, let $\mathbf{A}_{1,h} := p_h' \mathbf{A}_1 : X_1 \to X_{1,h}'$ where $p_h : X_{1,h} \to X_1$ is the canonical injection with adjoint $p_h' : X_1' \to X_{1,h}'$. Assume that

(i) the nonlinear operator $\mathbf{A}_{1,h} : X_1 \to X_{1,h}'$ is Lipschitz-continuous with a Lipschitz constant $\gamma_h > 0$, and for any $\tilde{\mathbf{t}} \in X_{1,h}$, the nonlinear operator $\Pi_{1,h}\mathbf{A}_{1,h}(\cdot + \tilde{\mathbf{t}}) : V_{1,h} \to V_{1,h}'$ is strongly monotone with a monotonicity constant $\alpha_h > 0$ independent of $\tilde{\mathbf{t}}$.

(ii) There exists $\beta_h > 0$ such that for all $v_h \in M_h$

$$\sup_{\substack{\boldsymbol{\tau}_h \in M_{1,h} \\ \boldsymbol{\tau}_h \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), v_h]}{\|\boldsymbol{\tau}_h\|_{M_1}} \geqslant \beta_h \|v_h\|_M \tag{11}$$

(iii) there exists $\beta_{1,h} > 0$ such that for all $\boldsymbol{\tau}_h \in \tilde{M}_{1,h}$

$$\sup_{\substack{\mathbf{s}_h \in X_{1,h} \\ \mathbf{s}_h \neq 0}} \frac{[\mathbf{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{X_1}} \geqslant \beta_{1,h} \|\boldsymbol{\tau}_h\|_{M_1} \tag{12}$$

Then, for each $(\mathbf{H}, \mathbf{G}, \mathbf{F}) \in X_1' \times M_1' \times M'$ there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ solution of (10). Moreover, there exists $C_h > 0$, independent of the solution, but depending on h, such that

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\|_{X_1 \times M_1 \times M} \leqslant C_h \{\|\mathbf{H}_h\| + \|\mathbf{G}_h\| + \|\mathbf{F}_h\| + \|\mathbf{A}_{1,h}(0)\|\}$$

where $\mathbf{H}_h := \mathbf{H}|_{X_{1,h}}$, $\mathbf{G}_h := \mathbf{G}|_{M_{1,h}}$, and $\mathbf{F}_h := \mathbf{F}|_{M_h}$.

Proof

See Theorem 3.2 in Reference [6] (see also References [15, Theorem 3], [16, Theorem 3.1], or [5, Theorem 4.2]). □

Finally, concerning the error analysis, we have the following result.

Theorem 2.3

Assume that all the hypotheses of both Theorems 2.1 and 2.2 are satisfied, and let $(\mathbf{t}, \boldsymbol{\sigma}, u) \in X_1 \times M_1 \times M$ and $(\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h) \in X_{1,h} \times M_{1,h} \times M_h$ be the unique solutions of (7) and (10), respectively. In addition, suppose that there exist positive constants $\tilde{\gamma}$, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\beta}_1$ such that

$\gamma_h \leqslant \tilde{\gamma}$, $\alpha_h \geqslant \tilde{\alpha}$, $\beta_h \geqslant \tilde{\beta}$, and $\beta_{1,h} \geqslant \tilde{\beta}_1$ for all h. Then, there exists $C>0$, independent of h, such that the following *Cea* error estimate holds:

$$\|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, u_h)\| \leqslant C \inf_{\substack{(\mathbf{s}_h, \boldsymbol{\tau}_h, v_h) \\ \in X_{1,h} \times M_{1,h} \times M_h}} \|(\mathbf{t}, \boldsymbol{\sigma}, u) - (\mathbf{s}_h, \boldsymbol{\tau}_h, v_h)\| \tag{13}$$

Proof
See Theorem 4.1 in Reference [6] (see also References [15, Theorem 5] or [16, Theorem 3.3]). □

2.3. Well posedness of the dual-mixed variational formulation

In what follows, we apply the abstract theory from the previous section to conclude that (6) has a unique solution. More precisely, as required by Theorem 2.1, we first establish the strong monotonicity and Lipschitz continuity of $\mathscr{A}_1$, and then prove that $\mathscr{B}$ and $\mathscr{B}_1$ satisfy suitable inf–sup conditions.

Lemma 2.1
The nonlinear operator $\mathscr{A}_1 : \mathscr{X}_1 \to \mathscr{X}_1'$ is strongly monotone and Lipschitz continuous, that is, there exist α, $\gamma>0$ such that

$$[\mathscr{A}_1(\mathbf{t}) - \mathscr{A}_1(\mathbf{r}), \mathbf{t} - \mathbf{r}] \geqslant \alpha \|\mathbf{t} - \mathbf{r}\|^2_{\mathscr{X}_1}$$

and

$$|[\mathscr{A}_1(\mathbf{t}) - \mathscr{A}_1(\mathbf{r}), \mathbf{s}]| \leqslant \gamma \|\mathbf{t} - \mathbf{r}\|_{\mathscr{X}_1} \|\mathbf{s}\|_{\mathscr{X}_1}$$

for all $\mathbf{t}, \mathbf{r}, \mathbf{s} \in \mathscr{X}_1 := [L^2(\Omega)]^{2\times 2}$.

Proof
It is a straightforward consequence of assumptions (1) and (2). □

The continuous inf–sup condition for $\mathscr{B}$ is proved next.

Lemma 2.2
There exists $\beta>0$ such that for all $(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \mathscr{M}$ there holds

$$\sup_{\substack{(\boldsymbol{\tau}, q) \in \mathscr{M}_1 \\ (\boldsymbol{\tau}, q) \neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})]}{\|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}} \geqslant \beta \|(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})\|_{\mathscr{M}}$$

Proof
Given $(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \mathscr{M}$, we first let $\boldsymbol{\tau}(\mathbf{v}) := \mathbf{e}(\mathbf{z})$ where $\mathbf{z} \in [H^1(\Omega)]^2$ is the unique weak solution of the boundary value problem: $-\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{v}$ in Ω, $\mathbf{z} = \mathbf{0}$ on Γ_{D}, $\mathbf{e}(\mathbf{z})\boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}. It follows that $\mathbf{div}\,\boldsymbol{\tau}(\mathbf{v}) = -\mathbf{v}$ in Ω, $\boldsymbol{\tau}(\mathbf{v}) = \boldsymbol{\tau}(\mathbf{v})^{\mathrm{t}}$ in Ω, and $\boldsymbol{\tau}(\mathbf{v})\boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}, which implies that

$$[\mathscr{B}(\boldsymbol{\tau}(\mathbf{v}), 0), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})] = \|\mathbf{v}\|^2_{[L^2(\Omega)]^2}$$

In addition, Korn's inequality and the corresponding continuous dependence result yield the existence of $\bar{C}>0$ such that $\|\boldsymbol{\tau}(\mathbf{v})\|_{H(\mathbf{div};\Omega)} \leqslant \bar{C}\|\mathbf{v}\|_{[L^2(\Omega)]^2}$. In this way, we find that

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \frac{[\mathscr{B}(\boldsymbol{\tau}(\mathbf{v}),0),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|\boldsymbol{\tau}(\mathbf{v})\|_{H(\mathbf{div};\Omega)}} \geqslant \frac{1}{\bar{C}}\|\mathbf{v}\|_{[L^2(\Omega)]} \tag{14}$$

Now, according to Lemma 4.4 in Reference [17] (see also Reference [2, Lemma 4.2]), there exists $\boldsymbol{\tau}(\boldsymbol{\delta})\in H(\mathbf{div};\Omega)$ such that $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\delta})=\mathbf{0}$ in Ω, $\frac{1}{2}(\boldsymbol{\tau}(\boldsymbol{\delta})^{\mathrm{t}}-\boldsymbol{\tau}(\boldsymbol{\delta}))=\boldsymbol{\delta}$ in Ω, and $\boldsymbol{\tau}(\boldsymbol{\delta})\boldsymbol{\nu}=\mathbf{0}$ on Γ_{N}. In addition, there exists $\tilde{C}>0$, independent of $\boldsymbol{\delta}$, such that $\|\boldsymbol{\tau}(\boldsymbol{\delta})\|_{H(\mathbf{div};\Omega)}\leqslant \tilde{C}\|\boldsymbol{\delta}\|_{[L^2(\Omega)]^{2\times 2}}$. Then, using that $-\boldsymbol{\delta}:\boldsymbol{\tau}(\boldsymbol{\delta})=\boldsymbol{\delta}:\boldsymbol{\delta}$, we obtain that $[\mathscr{B}(\boldsymbol{\tau}(\boldsymbol{\delta}),0),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]=\|\boldsymbol{\delta}\|^2_{[L^2(\Omega)]^{2\times 2}}$, and hence

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \frac{[\mathscr{B}(\boldsymbol{\tau}(\boldsymbol{\delta}),0),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|\boldsymbol{\tau}(\boldsymbol{\delta})\|_{H(\mathbf{div};\Omega)}} \geqslant \frac{1}{\tilde{C}}\|\boldsymbol{\delta}\|_{[L^2(\Omega)]^{2\times 2}} \tag{15}$$

On the other hand, given $\boldsymbol{\psi}\in[H^{-1/2}(\Gamma_{\mathrm{N}})]^2$, we let $\boldsymbol{\tau}(\boldsymbol{\psi}):=\mathbf{e}(\mathbf{z})$ where $\mathbf{z}\in[H^1(\Omega)]^2$ is the unique weak solution of the boundary value problem: $\mathbf{div}\,\mathbf{e}(\mathbf{z})=\mathbf{0}$ in Ω, $\mathbf{z}=\mathbf{0}$ on Γ_{D}, $\mathbf{e}(\mathbf{z})\boldsymbol{\nu}=\boldsymbol{\psi}$ on Γ_{N}. It follows that $\mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\psi})=\mathbf{0}$ in Ω, $\boldsymbol{\tau}(\boldsymbol{\psi})=\boldsymbol{\tau}(\boldsymbol{\psi})^{\mathrm{t}}$ in Ω, and $\boldsymbol{\tau}(\boldsymbol{\psi})\boldsymbol{\nu}=\boldsymbol{\psi}$ on Γ_{N}, whence we get

$$[\mathscr{B}(\boldsymbol{\tau}(\boldsymbol{\psi}),0),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]=\langle\boldsymbol{\psi},\boldsymbol{\lambda}\rangle_{\Gamma_{\mathrm{N}}}$$

In addition, similarly as before, there exists $\hat{C}>0$ such that $\|\boldsymbol{\tau}(\boldsymbol{\psi})\|_{H(\mathbf{div};\Omega)}\leqslant\hat{C}\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}$, and hence we deduce that

$$\begin{aligned}\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} &\geqslant \frac{|[\mathscr{B}(\boldsymbol{\tau}(\boldsymbol{\psi}),0),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]|}{\|\boldsymbol{\tau}(\boldsymbol{\psi})\|_{H(\mathbf{div};\Omega)}}\\ &\geqslant \frac{1}{\hat{C}}\frac{|\langle\boldsymbol{\psi},\boldsymbol{\lambda}\rangle_{\Gamma_{\mathrm{N}}}|}{\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}}\quad\forall\boldsymbol{\psi}\in[H^{-1/2}(\Gamma_{\mathrm{N}})]^2\end{aligned}$$

which gives

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_1\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \frac{1}{\hat{C}}\sup_{\substack{\boldsymbol{\psi}\in[H^{-1/2}(\Gamma_{\mathrm{N}})]^2\\ \boldsymbol{\psi}\neq 0}}\frac{|\langle\boldsymbol{\psi},\boldsymbol{\lambda}\rangle_{\Gamma_{\mathrm{N}}}|}{\|\boldsymbol{\psi}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}}=\frac{1}{\hat{C}}\|\boldsymbol{\lambda}\|_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2} \tag{16}$$

Finally, (14)–(16) provide the required inequality and complete the proof. □

In what follows, we let $\tilde{\mathscr{M}}_1$ be the null space of $\mathscr{B}$, that is

$$\tilde{\mathscr{M}}_1:=\{(\boldsymbol{\tau},q)\in\mathscr{M}_1:[\mathscr{B}(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]=0\ \forall(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})\in\mathscr{M}\}$$

Then, it is easy to see that $\tilde{\mathscr{M}}_1=\tilde{\mathscr{M}}_1^{\boldsymbol{\sigma}}\times L^2(\Omega)$, where

$$\tilde{\mathscr{M}}_1^{\boldsymbol{\sigma}}:=\{\boldsymbol{\tau}\in H(\mathbf{div};\Omega):\mathbf{div}\,\boldsymbol{\tau}=\mathbf{0}\text{ in }\Omega,\ \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathrm{t}}\text{ in }\Omega,\ \boldsymbol{\tau}\boldsymbol{\nu}=\mathbf{0}\text{ on }\Gamma_{\mathrm{N}}\}$$

Next, we define $H_0(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\}$ and note that $H(\mathbf{div};\Omega) = H_0(\mathbf{div};\Omega) \oplus \mathbb{R}\,\mathbf{I}$. This means that for any $\boldsymbol{\tau} \in H(\mathbf{div};\Omega)$ there exist unique $\boldsymbol{\tau}_0 \in H_0(\mathbf{div};\Omega)$ and $d := (1/2|\Omega|)\int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \in \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I}$, whence $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} = \|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\Omega)} + 2\,d^2\,|\Omega|$. In addition, we have the following lemmas.

Lemma 2.3

There exists $c_1 > 0$, depending only on Ω, such that

$$c_1\,\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} \leqslant \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \quad \forall\,\boldsymbol{\tau} \in H_0(\mathbf{div};\Omega) \tag{17}$$

and

$$c_1\,\|\boldsymbol{\tau}_0\|^2_{[L^2(\Omega)]^{2\times 2}} \leqslant \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \quad \forall \boldsymbol{\tau} \in H(\mathbf{div};\Omega) \tag{18}$$

Proof

For the proof of (17) we refer to Lemma 3.1 in Reference [8] or Proposition 3.1 of Chapter IV in Reference [9]. Then, given $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I} \in H(\mathbf{div};\Omega)$, with $\boldsymbol{\tau}_0 \in H_0(\mathbf{div};\Omega)$ and $d \in \mathbb{R}$, we note that (18) follows from (17) and the fact that $\boldsymbol{\tau}_0^{\mathrm{d}} = \boldsymbol{\tau}^{\mathrm{d}}$ and $\mathbf{div}(\boldsymbol{\tau}_0) = \mathbf{div}(\boldsymbol{\tau})$. □

Lemma 2.4

There exists $c_2 > 0$, depending only on Γ_{N} and Ω, such that

$$c_2\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} \leqslant \|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\,\Omega)} \quad \forall\,\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \quad \text{such that } \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_{\mathrm{N}} \tag{19}$$

Proof

Given $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\,\mathbf{I} \in H(\mathbf{div};\Omega)$, with $\boldsymbol{\tau}_0 \in H_0(\mathbf{div};\Omega)$ and $d \in \mathbb{R}$, and such that $\boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}, we note that $d\,\boldsymbol{\nu} = -\,\boldsymbol{\tau}_0\,\boldsymbol{\nu}$ on Γ_{N}, and hence

$$|d|\,\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2} = \|\boldsymbol{\tau}_0\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2} \leqslant \|\boldsymbol{\tau}_0\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma)]^2} \leqslant \|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\,\Omega)}$$

which yields

$$|d| \leqslant \frac{1}{\|\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}}\,\|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\,\Omega)}$$

This inequality and the fact that $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} = \|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\,\Omega)} + 2\,d^2|\Omega|$ imply (19). □

We now prove that $\mathscr{B}_1$ satisfies the continuous inf–sup condition on $\tilde{\mathscr{M}}_1$.

Lemma 2.5

There exists $\beta_1 > 0$ such that for all $(\boldsymbol{\tau}, q) \in \tilde{\mathscr{M}}_1$ there holds

$$\sup_{\substack{\mathbf{s}\in\mathscr{X}_1\\ \mathbf{s}\neq 0}} \frac{[\mathscr{B}_1(\mathbf{s}),\,(\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{\mathscr{X}_1}} \geqslant \beta_1\,\|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1} \tag{20}$$

Proof

Let $(\boldsymbol{\tau}, q) \in \tilde{\mathscr{M}}_1$ such that $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \geqslant \|q\|_{L^2(\Omega)}$. Since $\operatorname{tr}(\boldsymbol{\tau}^{\mathrm{d}}) = 0$, we easily obtain that $[\mathscr{B}_1(-\,\boldsymbol{\tau}^{\mathrm{d}}),\,\boldsymbol{\tau}] = \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}}$. In addition, according to the definition of $\tilde{\mathscr{M}}_1^{\boldsymbol{\sigma}}$ there hold

$\mathbf{div}(\boldsymbol{\tau}) = 0$ in Ω and $\boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}, whence we deduce from (18) and (19) that there exists $C_1 > 0$, depending only on Ω, such that

$$\|\boldsymbol{\tau}^{\mathrm{d}}\|_{[L^2(\Omega)]^{2\times 2}} \geqslant C_1 \, \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \tag{21}$$

Therefore, assuming that $\boldsymbol{\tau}^{\mathrm{d}} \neq \mathbf{0}$, we can write

$$\sup_{\substack{\mathbf{s}\in\mathscr{X}_1\\ \mathbf{s}\neq 0}} \frac{[\mathscr{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{\mathscr{X}_1}} \geqslant \frac{[\mathscr{B}_1(-\boldsymbol{\tau}^{\mathrm{d}}), (\boldsymbol{\tau}, q)]}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{\mathscr{X}_1}} = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{[L^2(\Omega)]^{2\times 2}}$$

$$\geqslant C_1 \, \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \geqslant \frac{C_1}{2} \, \|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}$$

Now, if $\boldsymbol{\tau}^{\mathrm{d}} = \mathbf{0}$ then inequality (21) and the fact that $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \geqslant \|q\|_{L^2(\Omega)}$ yield $(\boldsymbol{\tau}, q) = (\mathbf{0}, 0)$, and thus estimate (20) is trivially satisfied.

On the other hand, let $(\boldsymbol{\tau}, q) \in \tilde{\mathscr{M}}_1$ such that $\|q\|_{L^2(\Omega)} \geqslant \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$. A simple computation shows that

$$[\mathscr{B}_1(q\mathbf{I} + \boldsymbol{\tau}), (\boldsymbol{\tau}, q)] = 2\|q\|^2_{L^2(\Omega)} - \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} \geqslant \|q\|^2_{L^2(\Omega)} \tag{22}$$

and hence, assuming that $q\,\mathbf{I} + \boldsymbol{\tau} \neq \mathbf{0}$, we find that

$$\sup_{\substack{\mathbf{s}\in\mathscr{X}_1\\ \mathbf{s}\neq 0}} \frac{[\mathscr{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{\mathscr{X}_1}} \geqslant \frac{[\mathscr{B}_1(q\mathbf{I} + \boldsymbol{\tau}), (\boldsymbol{\tau}, q)]}{\|q\mathbf{I} + \boldsymbol{\tau}\|_{\mathscr{X}_1}} \geqslant C_2\|q\|_{L^2(\Omega)} \geqslant \frac{C_2}{2} \, \|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}$$

Similarly, if $q\mathbf{I} + \boldsymbol{\tau} = \mathbf{0}$ then inequality (22) and the fact that $\|q\|_{L^2(\Omega)} \geqslant \|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$ imply that $(\boldsymbol{\tau}, q) = (\mathbf{0}, 0)$, and hence (20) follows straightforwardly again. □

The unique solvability of (4) (equivalently (6)) is established now.

Theorem 2.4
There exists a unique $(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \gamma, \xi)) \in \mathscr{X}_1 \times \mathscr{M}_1 \times \mathscr{M}$ solution of (4). Moreover, there exists $C > 0$ such that

$$\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \gamma, \xi))\| \leqslant C\{\|\mathscr{G}\|_{\mathscr{M}'} + \|\mathscr{N}(\mathbf{0})\|_{[L^2(\Omega)]^{2\times 2}}\} \tag{23}$$

where $\|\mathscr{G}\|_{\mathscr{M}'} \leqslant \|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}$.

Proof
It follows from Lemmata 2.1, 2.2, 2.5, and a direct application of the abstract result given by Theorem 2.1. □

3. THE DUAL-MIXED FINITE ELEMENT SCHEME

We now let $\{\mathscr{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$, made up of triangles T of diameter h_T, such that $h := \max\{h_T : T \in \mathscr{T}_h\}$, and assume that all the points in $\bar{\Gamma}_{\mathrm{D}} \cap \bar{\Gamma}_{\mathrm{N}}$

become vertices of $\mathcal{T}_h$ for all $h>0$. Also, for reasons that will become clear below, we introduce an independent partition $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_m\}$ of Γ_N and denote $\tilde{h} := \max\{|\tilde{\Gamma}_j| : j \in \{1, \dots, m\}\}$. Next, we let $\mathcal{X}_{1,h}$, $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, $\mathcal{M}^{p}_{1,h}$, $\mathcal{M}^{\mathbf{u}}_{h}$, $\mathcal{M}^{\boldsymbol{\gamma}}_{h}$, and $\mathcal{M}^{\boldsymbol{\xi}}_{\tilde{h}}$ be finite element subspaces for the unknowns $\mathbf{t}$, $\boldsymbol{\sigma}$, p, $\mathbf{u}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\xi}$, respectively, and define the product spaces $\mathcal{M}_{1,h} := \mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \times \mathcal{M}^{p}_{1,h}$ and $\mathcal{M}_{h,\tilde{h}} := \mathcal{M}^{\mathbf{u}}_{h} \times \mathcal{M}^{\boldsymbol{\gamma}}_{h} \times \mathcal{M}^{\boldsymbol{\xi}}_{\tilde{h}}$. Then, the Galerkin scheme associated with (4) reads as follows: Find $(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in \mathcal{X}_{1,h} \times \mathcal{M}_{1,h} \times \mathcal{M}_{h,\tilde{h}}$ such that

$$\begin{aligned} \int_\Omega \mathcal{N}(\mathbf{t}_h) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}_h : \mathbf{s} + \int_\Omega p_h \operatorname{tr}(\mathbf{s}) &= 0 \\ -\int_\Omega \mathbf{t}_h : \boldsymbol{\tau} + \int_\Omega q_h \operatorname{tr}(\mathbf{t}_h) - \int_\Omega \mathbf{u}_h \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\gamma}_h : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\, \mathbf{v}, \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_N} &= 0 \\ -\int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}_h - \int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma}_h - \langle \boldsymbol{\sigma}_h \mathbf{v}, \boldsymbol{\lambda} \rangle_{\Gamma_N} &= \mathcal{G}(\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \end{aligned} \tag{24}$$

for all $(\mathbf{s}, (\boldsymbol{\tau}, q), (\mathbf{v}, \boldsymbol{\delta}, \boldsymbol{\lambda})) \in \mathcal{X}_{1,h} \times \mathcal{M}_{1,h} \times \mathcal{M}_{h,\tilde{h}}$.

Throughout the rest of this section we show that (24) satisfies the hypotheses of Theorem 2.2, which certainly depends on the specific finite element subspaces to be utilized. Actually, the explicit definition of these subspaces will be derived from the need of verifying the discrete inf–sup conditions for $\mathcal{B}$ and $\mathcal{B}_1$. To this end, our analysis follows some of the arguments employed in Section 2 and extend them, as much as possible, to the present discrete case.

We begin with the discrete inf–sup condition for $\mathcal{B}$ noting that, given $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathcal{M}_{h,\tilde{h}}$, there holds

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathcal{M}_{1,h}\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau}, q)\|_{\mathcal{M}_1}} \geqslant \sup_{\substack{\boldsymbol{\tau}\in\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}\\ \boldsymbol{\tau}\neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \tag{25}$$

where, according to the definition of $\mathcal{B}$,

$$[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})] = -\int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau} - \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau} - \langle \boldsymbol{\tau}\mathbf{v}, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}$$

Then we observe that, except for the term involving $\boldsymbol{\lambda}_{\tilde{h}}$, the right-hand side of (25) coincides with the expression in Reference [7] for which the well known PEERS finite element subspace satisfies the corresponding discrete inf–sup condition. Motivated by this fact, in what follows we let $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h} \times \mathcal{M}^{\mathbf{u}}_{h} \times \mathcal{M}^{\boldsymbol{\gamma}}_{h}$ be exactly the PEERS space introduced in Reference [7]. More precisely, we define

$$\mathcal{M}^{\boldsymbol{\sigma}}_{1,h} := \{\boldsymbol{\tau}_h \in H(\mathbf{div}; \Omega) : \boldsymbol{\tau}_h|_T \in [\mathcal{RT}_0(T)^{\mathrm{t}}]^2 \oplus [\mathbb{P}_0(T)\, \mathbf{curl}^{\mathrm{t}}\, b_T]^2 \ \forall T \in \mathcal{T}_h\} \tag{26}$$

$$\mathcal{M}^{\mathbf{u}}_{h} := \{\mathbf{v}_h \in [L^2(\Omega)]^2 : \mathbf{v}_h|_T \in [\mathbb{P}_0(T)]^2 \ \forall T \in \mathcal{T}_h\} \tag{27}$$

and

$$\mathscr{M}_h^{\gamma} := \left\{ \begin{pmatrix} 0 & \delta_h \\ -\delta_h & 0 \end{pmatrix} \in [H^1(\Omega)]^{2\times 2} : \delta_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathscr{T}_h \right\} \tag{28}$$

where $\mathscr{RT}_0(T)$ is the local Raviart–Thomas space of order 0 (cf. References [9, 18]), b_T is the usual cubic bubble function on $T \in \mathscr{T}_h$, and $\mathbf{curl}^{\mathrm{t}}\, b_T := (\partial b_T/\partial x_2, -\partial b_T/\partial x_1)$. Hereafter, given an integer $\ell \geqslant 0$ and a subset S of $\mathbb{R}^2$, $\mathbb{P}_\ell(S)$ stands for the space of polynomials defined in S of total degree $\leqslant \ell$.

The following result provides a preliminary estimate.

Lemma 3.1
There exists $\widehat{C}>0$, independent of h and $\tilde{h}$, such that for all $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}$ there holds

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_{1,h}\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}} \geqslant \widehat{C}\|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}} \tag{29}$$

Proof
We recall from Lemma 4.4 in Reference [7] that given $(\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathscr{M}_h^{\mathbf{u}} \times \mathscr{M}_h^{\gamma}$ there exists $\widehat{\boldsymbol{\tau}}_h \in \mathscr{M}_{1,h}^{\boldsymbol{\sigma}}$, $\widehat{\boldsymbol{\tau}}_h \neq \mathbf{0}$, such that

$$[\mathscr{B}(\widehat{\boldsymbol{\tau}}_h, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, 0)] \geqslant \widehat{C}\|\widehat{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\Omega)}\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}}$$

with a constant $\widehat{C}>0$, independent of h and $(\mathbf{v}_h, \boldsymbol{\delta}_h)$. Furthermore, $\widehat{\boldsymbol{\tau}}_h$ can be chosen so that $\widehat{\boldsymbol{\tau}}_h \boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}, and hence

$$\sup_{\substack{\boldsymbol{\tau}\in\mathscr{M}_{1,h}^{\boldsymbol{\sigma}}\\\boldsymbol{\tau}\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geqslant \frac{[\mathscr{B}(\widehat{\boldsymbol{\tau}}_h, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|\widehat{\boldsymbol{\tau}}_h\|_{H(\mathbf{div};\Omega)}} \geqslant \widehat{C}\, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}}$$

which, together with (25), yields (29). □

In order to continue the analysis of the discrete inf–sup condition for $\mathscr{B}$, we now introduce the continuous piecewise linear functions on Γ_{N} vanishing at the end points of Γ_{N} as the finite element subspace of $[H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2$, that is we set

$$\mathscr{M}_{\tilde{h}}^{\boldsymbol{\xi}} := \{\boldsymbol{\lambda}_{\tilde{h}} \in [H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2 : \boldsymbol{\lambda}_{\tilde{h}}|_{\tilde{\Gamma}_j} \in [\mathbb{P}_1(\tilde{\Gamma}_j)]^2 \ \forall j \in \{1, \ldots, m\}\} \tag{30}$$

which, as it is well known, satisfies the following approximation property (see References [19, 20]):

$(AP_{\tilde{h}}^{\boldsymbol{\xi}})$ for each $t \in (-\frac{1}{2}, \frac{3}{2}]$ and for each $\boldsymbol{\lambda} \in [H^t(\Gamma_{\mathrm{N}})]^2 \cap [H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2$ there exists $\boldsymbol{\lambda}_{\tilde{h}} \in \mathscr{M}_{\tilde{h}}^{\boldsymbol{\xi}}$ such that

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2} \leqslant C\, \tilde{h}^{t-1/2}\, \|\boldsymbol{\lambda}\|_{[H^t(\Gamma_{\mathrm{N}})]^2}$$

Next, we proceed similarly to the analysis in Reference [21] and introduce the subspace of $[H^{-1/2}(\Gamma_{\mathrm{N}})]^2$ given by the piecewise constant functions. In other words, if $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$

is the partition on Γ_N induced by the triangulation $\mathscr{T}_h$, we define

$$H_h^{-1/2} := \{\psi_h \in [L^2(\Gamma_N)]^2 : \psi_h|_{\Gamma_j} \in [\mathbb{P}_0(\Gamma_j)]^2 \ \forall j \in \{1, \ldots, n\}\}$$

which satisfies the following approximation property (see References [19, 20]):

$(AP_h^{-1/2})$ for each $s \in (-\frac{1}{2}, \frac{1}{2}]$ and for each $\psi \in [H^s(\Gamma_N)]^2$ there exists $\psi_h \in H_h^{-1/2}$ such that

$$\|\psi - \psi_h\|_{[H^{-1/2}(\Gamma_N)]^2} \leqslant C h^{s+1/2} \|\psi\|_{[H^s(\Gamma_N)]^2}$$

Also, we assume that $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ and $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_m\}$ are uniformly regular, which means that there exist $C, \tilde{C}>0$, independent of h and $\tilde{h}$, such that $|\Gamma_j| \geqslant C\, h$ for all $j \in \{1, \ldots, n\}$ and $|\tilde{\Gamma}_j| \geqslant \tilde{C}\, \tilde{h}$ for all $j \in \{1, \ldots, m\}$, for all $h, \tilde{h}>0$. These conditions yield the inverse inequalities for the spaces $H_h^{-1/2}$ and $\mathscr{M}_{\tilde{h}}^{\xi}$ (see References [19, 20]), respectively, that is, for any real numbers s and t with $-\frac{1}{2} \leqslant s \leqslant t \leqslant 0$, there exists $C>0$ such that

$$\|\psi_h\|_{[H^t(\Gamma_N)]^2} \leqslant C\, h^{s-t}\, \|\psi_h\|_{[H^s(\Gamma_N)]^2} \quad \forall\, \psi_h \in H_h^{-1/2} \tag{31}$$

and for any real numbers s and t with $0 \leqslant s \leqslant t \leqslant 1$, there exists $\tilde{C}>0$ such that

$$\|\lambda_{\tilde{h}}\|_{[H^t(\Gamma_N)]^2} \leqslant \tilde{C}\, \tilde{h}^{s-t}\, \|\lambda_{\tilde{h}}\|_{[H^s(\Gamma_N)]^2} \quad \forall\, \lambda_{\tilde{h}} \in \mathscr{M}_{\tilde{h}}^{\xi} \tag{32}$$

The following lemma establishes a second preliminary estimate.

Lemma 3.2

There exist $C_0 \in]0, 1[$, $C_1 > 0$, independent of h and $\tilde{h}$, such that for all $h \leqslant C_0\, \tilde{h}$ and for all $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}$ there holds

$$\sup_{\substack{(\boldsymbol{\tau},q)\in\mathscr{M}_{1,h}\\(\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}} \geqslant C_1 \|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} - \|\boldsymbol{\delta}_h\|_{\mathscr{R}} \tag{33}$$

Proof

We proceed as in the last part of the proof of Lemma 2.2. Hence, given $\psi_h \in H_h^{-1/2}$ we let $\mathbf{z} \in [H_{\Gamma_D}^1(\Omega)]^2$ be the unique weak solution of the boundary value problem:

$$\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{0} \quad \text{in } \Omega, \quad \mathbf{z} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \mathbf{e}(\mathbf{z})\,\mathbf{v} = \psi_h \quad \text{on } \Gamma_N$$

Since $H_h^{-1/2} \subseteq [L^2(\Gamma_N)]^2$, the classical regularity results in Reference [22] (see also Reference [23]) imply that $\mathbf{z} \in [H^{1+\delta}(\Omega)]^2$ and $\|\mathbf{z}\|_{[H^{1+\delta}(\Omega)]^2} \leqslant C \|\psi_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2}$, where $\delta := \min\{\frac{1}{2}, \pi/2\omega\}$ and $\omega \in]0, 2\pi[$ is the largest interior angle of Ω. Furthermore, the usual continuous dependence result establishes that $\|\mathbf{z}\|_{[H^1(\Omega)]^2} \leqslant C \|\psi_h\|_{[H^{-1/2}(\Gamma_N)]^2}$. Thus, we let $\boldsymbol{\tau}(\psi_h) := \mathbf{e}(\mathbf{z}) \in [H^{\delta}(\Omega)]^{2\times 2}$ and observe that $\mathbf{div}\,\boldsymbol{\tau}(\psi_h) = \mathbf{0}$ in Ω and $\boldsymbol{\tau}(\psi_h)\mathbf{v} = \psi_h$ on Γ_N. In addition, according to the previous estimates for $\mathbf{z}$, we obtain that

$$\|\boldsymbol{\tau}(\psi_h)\|_{H(\mathbf{div};\Omega)} = \|\boldsymbol{\tau}(\psi_h)\|_{[L^2(\Omega)]^{2\times 2}} \leqslant \tilde{C} \|\psi_h\|_{[H^{-1/2}(\Gamma_N)]^2} \tag{34}$$

and

$$\|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[H^\delta(\Omega)]^{2\times 2}} \leqslant C\|\boldsymbol{\psi}_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2} \tag{35}$$

Next, we let $\bar{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h} := \{\boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \boldsymbol{\tau}_h|_T \in [\mathscr{R}\mathscr{T}_0(T)^{\mathrm{t}}]^2 \ \forall T \in \mathscr{T}_h\}$ be the usual Raviart–Thomas subspace, which is certainly contained in $\mathscr{M}^{\boldsymbol{\sigma}}_{1,h}$, and consider the equilibrium interpolation operator $\mathscr{E}_h : [H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega) \to \bar{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h}$ (see References [9, 18], [24, Theorem 3.1]). It is well known, as proved by Theorem 6.3 in Reference [18] and Theorem 3.4 in Reference [24], that $\mathscr{E}_h$ satisfies the following approximation property

$$\|\boldsymbol{\tau} - \mathscr{E}_h(\boldsymbol{\tau})\|_{[L^2(\Omega)]^{2\times 2}} \leqslant C\, h^\delta \|\boldsymbol{\tau}\|_{[H^\delta(\Omega)]^{2\times 2}} \quad \forall \boldsymbol{\tau} \in [H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega) \tag{36}$$

Moreover, there hold $\mathbf{div}\,\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h)) = \mathbf{div}\,\boldsymbol{\tau}(\boldsymbol{\psi}_h) = \mathbf{0}$ in Ω and $\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\boldsymbol{\nu} = \boldsymbol{\tau}(\boldsymbol{\psi}_h)\boldsymbol{\nu} = \boldsymbol{\psi}_h$ on Γ_N. Then, using (36), (34), and (35), we obtain

$$\begin{aligned}
\|\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)} &\leqslant \|\boldsymbol{\tau}(\boldsymbol{\psi}_h) - \mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[L^2(\Omega)]^{2\times 2}} \\
&\leqslant Ch^\delta \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[H^\delta(\Omega)]^{2\times 2}} + \|\boldsymbol{\tau}(\boldsymbol{\psi}_h)\|_{[L^2(\Omega)]^{2\times 2}} \\
&\leqslant Ch^\delta \|\boldsymbol{\psi}_h\|_{[H^{-1/2+\delta}(\Gamma_N)]^2} + \tilde{C}\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}
\end{aligned}$$

which, applying inverse inequality (31), yields

$$\|\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)} \leqslant C\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}$$

Therefore, given $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}$, we find that

$$\begin{aligned}
\sup_{\substack{\boldsymbol{\tau} \in \mathscr{M}^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} &\geqslant \frac{|[\mathcal{B}(\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h)), 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]|}{\|\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)}} \\
&= \frac{|-\langle \boldsymbol{\psi}_h, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N} - \int_\Omega \boldsymbol{\delta}_h : \mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))|}{\|\mathscr{E}_h(\boldsymbol{\tau}(\boldsymbol{\psi}_h))\|_{H(\mathbf{div};\Omega)}} \\
&\geqslant \frac{1}{C} \frac{|\langle \boldsymbol{\psi}_h, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} - \|\boldsymbol{\delta}_h\|_{\mathscr{R}}
\end{aligned}$$

for all $\boldsymbol{\psi}_h \in H_h^{-1/2}$, and hence,

$$\sup_{\substack{\boldsymbol{\tau} \in \mathscr{M}^{\boldsymbol{\sigma}}_{1,h} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathcal{B}(\boldsymbol{\tau}, 0), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geqslant \frac{1}{C} \sup_{\substack{\boldsymbol{\psi}_h \in H_h^{-1/2} \\ \boldsymbol{\psi}_h \neq 0}} \frac{|\langle \boldsymbol{\psi}_h, \boldsymbol{\lambda}_{\tilde{h}}\rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} - \|\boldsymbol{\delta}_h\|_{\mathscr{R}} \tag{37}$$

On the other hand, since the normal trace on Γ_N is well defined and continuous from $[H^\delta(\Omega)]^{2\times 2} \cap H(\mathbf{div};\Omega)$ onto $[H^{-1/2+\delta}(\Gamma_N)]^2$ for $\delta \neq \frac{1}{2}$ (see Theorem 2.4 and the corresponding remark in Reference [24]), we can apply the vector version of Lemma 3.3 in Reference

[21], making use of the approximation property $(AP_h^{-1/2})$ and the inverse inequality (32), to deduce that there exist $C_0 \in]0,1[$, $\bar{C}>0$, independent of h and $\tilde{h}$, such that for all $h \leqslant C_0 \tilde{h}$ there holds

$$\sup_{\substack{\psi_h \in H_h^{-1/2} \\ \psi_h \neq 0}} \frac{|\langle \boldsymbol{\psi}_h, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N}|}{\|\boldsymbol{\psi}_h\|_{[H^{-1/2}(\Gamma_N)]^2}} \geqslant \bar{C} \|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} \tag{38}$$

In this way, (25), (37) and (38) yield (33). □

We are now in a position to establish the discrete inf–sup condition for $\mathscr{B}$.

Lemma 3.3

Let $C_0 \in]0,1[$ be the constant given in Lemma 3.2. Then there exists $\bar{\beta}>0$, independent of h and $\tilde{h}$, such that for all $h \leqslant C_0 \tilde{h}$ and for all $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}$ there holds

$$\sup_{\substack{(\boldsymbol{\tau},q)\in \mathscr{M}_{1,h} \\ (\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \bar{\beta} \|(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})\|_{\mathscr{M}} \tag{39}$$

Proof

Let $C_1>0$ be the constant in (33) (cf. Lemma 3.2) and let $(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}$. We separate the proof in two possible cases. If $\|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}} \leqslant (C_1/2)\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}$, then (33) yields

$$\sup_{\substack{(\boldsymbol{\tau},q)\in \mathscr{M}_{1,h} \\ (\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \frac{C_1}{2} \|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2} \geqslant \bar{\beta}_1 \, \|(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})\|_{\mathscr{M}}$$

with $\bar{\beta}_1 := \min\{\frac{1}{2}, C_1/4\}$. On the other hand, if $\|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}} \geqslant (C_1/2)\|\boldsymbol{\lambda}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_N)]^2}$, then we deduce from (29) that

$$\sup_{\substack{(\boldsymbol{\tau},q)\in \mathscr{M}_{1,h} \\ (\boldsymbol{\tau},q)\neq 0}} \frac{[\mathscr{B}(\boldsymbol{\tau},q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})]}{\|(\boldsymbol{\tau},q)\|_{\mathscr{M}_1}} \geqslant \widehat{C} \, \|(\mathbf{v}_h, \boldsymbol{\delta}_h)\|_{[L^2(\Omega)]^2\times\mathscr{R}} \geqslant \bar{\beta}_2 \, \|(\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})\|_{\mathscr{M}}$$

with $\bar{\beta}_2 := \min\{\widehat{C}C_1/4, \widehat{C}/2\}$. Finally, (39) follows by choosing $\bar{\beta} := \min\{\bar{\beta}_1, \bar{\beta}_2\}$. □

Our next goal is the discrete inf–sup condition for $\mathscr{B}_1$. For this purpose, we first identify the discrete kernel of the operator $\mathscr{B}$, that is

$$\tilde{\mathscr{M}}_{1,h} := \{(\boldsymbol{\tau},q) \in \mathscr{M}_{1,h} : [\mathscr{B}(\boldsymbol{\tau},q), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})] = 0 \ \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}}\}$$

which becomes

$$\tilde{\mathscr{M}}_{1,h} := \tilde{\mathscr{M}}_{1,h}^{\boldsymbol{\sigma}} \times \mathscr{M}_{1,h}^{p}$$

where

$$\tilde{\mathscr{M}}_{1,h}^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau} \in \mathscr{M}_{1,h}^{\boldsymbol{\sigma}} : \int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau} + \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\lambda}_{\tilde{h}} \rangle_{\Gamma_N} = 0 \ \ \forall (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}) \in \mathscr{M}_{h,\tilde{h}} \right\}$$

In addition, since $(\mathbf{div}\,\boldsymbol{\tau})|_T$ and $\mathbf{v}_h|_T$ are constant vectors for each $\boldsymbol{\tau}\in\mathscr{M}^{\boldsymbol{\sigma}}_{1,h}$ and for each $\mathbf{v}_h\in\mathscr{M}^{\mathbf{u}}_h$, we find that $\mathbf{div}\,\boldsymbol{\tau}$ vanishes in Ω, and hence

$$\tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h} := \left\{\boldsymbol{\tau}\in\mathscr{M}^{\boldsymbol{\sigma}}_{1,h} : \mathbf{div}\,\boldsymbol{\tau}=\mathbf{0}\text{ in } \Omega,\ \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}=0\ \ \forall\,\boldsymbol{\delta}_h\in\mathscr{M}^{\gamma}_h,\ \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\lambda}_{\tilde h}\rangle_{\Gamma_{\mathrm{N}}}=0\,\forall\,\boldsymbol{\lambda}_{\tilde h}\in\mathscr{M}^{\boldsymbol{\xi}}_{\tilde h}\right\}$$

Moreover, according to the definitions of $\mathscr{M}^{\boldsymbol{\sigma}}_{1,h}$ (cf. (26)) and $\mathscr{RT}_0(T)$ (see References [9, 18]), taking into account only the free-divergence property of the elements in $\tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h}$, and noting that $\mathbf{div}\,\mathbf{curl}=0$, it follows that

$$\tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h}\subseteq\{\boldsymbol{\tau}_h\in H(\mathbf{div};\Omega) : \boldsymbol{\tau}_h|_T\in[\mathbb{P}_0(T)]^{2\times 2}\oplus[\mathbb{P}_0(T)\,\mathbf{curl}^{\mathrm{t}}\,b_T]^2\ \ \forall\,T\in\mathscr{T}_h\} \tag{40}$$

At this point, we need to establish the discrete analogue of Lemma 2.4, which, similarly to the continuous case (cf. Lemma 2.5), will serve to show that the bilinear form $\mathscr{B}_1$ satisfies the discrete inf–sup condition on $\tilde{\mathscr{M}}_{1,h}$.

Lemma 3.4

Let $V_{h,\tilde h}:=\{\boldsymbol{\tau}\in\mathscr{M}^{\boldsymbol{\sigma}}_{1,h} : \langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\lambda}_{\tilde h}\rangle_{\Gamma_{\mathrm{N}}}=0\ \ \forall\,\boldsymbol{\lambda}_{\tilde h}\in\mathscr{M}^{\boldsymbol{\xi}}_{\tilde h}\}$. Then there exist $C_2, h_0>0$, independent of h and $\tilde h$, such that for each $\tilde h\leqslant h_0$ there holds:

$$C_2\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)}\leqslant\|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\,\Omega)}\quad\forall\,\boldsymbol{\tau}\in V_{h,\tilde h} \tag{41}$$

Proof

Given $\boldsymbol{\tau}\in V_{h,\tilde h}$ we first write $\boldsymbol{\tau}=\boldsymbol{\tau}_0+d\,\mathbf{I}$ with $\boldsymbol{\tau}_0\in H_0(\mathbf{div};\Omega)$ and $d\in\mathbb{R}$. In addition, we apply $(AP^{\boldsymbol{\xi}}_{\tilde h})$ with $t=3/2$ to each $\boldsymbol{\chi}\in[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2\subseteq[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2$ and deduce the existence of $\boldsymbol{\chi}_{\tilde h}\in\mathscr{M}^{\boldsymbol{\xi}}_{\tilde h}$ such that

$$\|\boldsymbol{\chi}-\boldsymbol{\chi}_{\tilde h}\|_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2}\leqslant C\,\tilde h\,\|\boldsymbol{\chi}\|_{[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2} \tag{42}$$

Then, according to the definition of $V_{h,\tilde h}$, we can write

$$\langle d\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}=\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}-\langle\boldsymbol{\tau}_0\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}=\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{\chi}-\boldsymbol{\chi}_{\tilde h}\rangle_{\Gamma_{\mathrm{N}}}-\langle\boldsymbol{\tau}_0\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}$$

which, using (42) and the continuous imbedding of $[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2$ into $[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2$, yields

$$\begin{aligned}
|\langle d\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}| &\leqslant \|\boldsymbol{\tau}\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}\,\|\boldsymbol{\chi}-\boldsymbol{\chi}_{\tilde h}\|_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2}+\|\boldsymbol{\tau}_0\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}\,\|\boldsymbol{\chi}\|_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2}\\
&\leqslant \tilde C\{\tilde h\|\boldsymbol{\tau}\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma)]^2}+\|\boldsymbol{\tau}_0\boldsymbol{\nu}\|_{[H^{-1/2}(\Gamma)]^2}\}\|\boldsymbol{\chi}\|_{[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2}\\
&\leqslant \tilde C\{\tilde h\|\boldsymbol{\tau}\|_{H(\mathbf{div};\,\Omega)}+\|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\,\Omega)}\}\|\boldsymbol{\chi}\|_{[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2}
\end{aligned}$$

where $\tilde C>0$ is independent of h and $\tilde h$. In this way, we have shown that

$$\frac{|\langle d\boldsymbol{\nu},\boldsymbol{\chi}\rangle_{\Gamma_{\mathrm{N}}}|}{\|\boldsymbol{\chi}\|_{[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2}}\leqslant\tilde C\{\tilde h\,\|\boldsymbol{\tau}\|_{H(\mathbf{div};\,\Omega)}+\|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\,\Omega)}\}\quad\forall\,\boldsymbol{\chi}\in[H^{3/2}_{00}(\Gamma_{\mathrm{N}})]^2,\quad\boldsymbol{\chi}\neq 0$$

which, noting that $[H^{-3/2}(\Gamma_N)]^2$ is the dual of $[H_{00}^{3/2}(\Gamma_N)]^2$, leads to

$$|d| \leqslant \frac{1}{\|\mathbf{v}\|_{[H^{-3/2}(\Gamma_N)]^2}} \tilde{C}\,\{\tilde{h}\,\|\boldsymbol{\tau}\|_{H(\mathbf{div};\,\Omega)} + \|\boldsymbol{\tau}_0\|_{H(\mathbf{div};\,\Omega)}\}$$

This inequality and the fact that $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\,\Omega)} = \|\boldsymbol{\tau}_0\|^2_{H(\mathbf{div};\,\Omega)} + 2d^2|\Omega|$ imply (41) for each $\tilde{h} \leqslant h_0$, with h_0 sufficiently small. We omit further details. □

As a consequence of (18) and (41), and noting that $\mathbf{div}\,\boldsymbol{\tau} = \mathbf{0}$ in Ω for each $\boldsymbol{\tau} \in \tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h}$, we obtain the discrete analogue of (21), that is there exists $\tilde{C}_1 > 0$, independent of h and $\tilde{h}$, such that for each $\tilde{h} \leqslant h_0$ there holds

$$\|\boldsymbol{\tau}^{\mathrm{d}}\|_{[L^2(\Omega)]^{2\times 2}} \geqslant \tilde{C}_1\,\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)} \quad \forall\,\boldsymbol{\tau} \in \tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h} \tag{43}$$

Hence, in order to extend the arguments yielding the continuous inf–sup condition for $\mathscr{B}_1$ (cf. Lemma 2.5) to the present discrete case, we only require that

$$\boldsymbol{\tau}^{\mathrm{d}} \text{ and } (q\mathbf{I} + \boldsymbol{\tau}) \in \mathscr{X}_{1,h} \quad \forall\,(\boldsymbol{\tau}, q) \in \tilde{\mathscr{M}}_{1,h} := \tilde{\mathscr{M}}^{\boldsymbol{\sigma}}_{1,h} \times \mathscr{M}^{p}_{1,h} \tag{44}$$

Now, since there is no specific requirement on $\mathscr{M}^p_{1,h}$, we can take this finite element subspace as the simplest one of $L^2(\Omega)$, that is

$$\mathscr{M}^p_{1,h} := \{q_h \in L^2(\Omega) : q_h|_T \in \mathbb{P}_0(T) \;\; \forall T \in \mathscr{T}_h\} \tag{45}$$

It follows from (40) that (44) is satisfied if the finite element subspace for the unknown $\mathbf{t} \in [L^2(\Omega)]^{2\times 2}$ is defined as follows:

$$\mathscr{X}_{1,h} := \{\mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \mathbf{s}|_T \in \mathscr{X}(T) \;\; \forall\,T \in \mathscr{T}_h\} \tag{46}$$

where

$$\mathscr{X}(T) := [\mathbb{P}_0(T)]^{2\times 2} \oplus [\mathbb{P}_0(T)\,\mathbf{curl}^{\mathrm{t}}\,b_T]^2 \oplus ([\mathbb{P}_0(T)\,\mathbf{curl}^{\mathrm{t}}\,b_T]^2)^{\mathrm{d}} \tag{47}$$

Lemma 3.5
Let $h_0 > 0$ be the constant given in Lemma 3.4. Then there exists $\bar{\beta}_1 > 0$, independent of h and $\tilde{h}$, such that for all $\tilde{h} \leqslant h_0$ and for all $(\boldsymbol{\tau}, q) \in \tilde{\mathscr{M}}_{1,h}$ there holds

$$\sup_{\substack{\mathbf{s}\in\mathscr{X}_{1,h}\\ \mathbf{s}\neq 0}} \frac{[\mathcal{B}_1(\mathbf{s}), (\boldsymbol{\tau}, q)]}{\|\mathbf{s}\|_{\mathscr{X}_1}} \geqslant \bar{\beta}_1\,\|(\boldsymbol{\tau}, q)\|_{\mathscr{M}_1}$$

Proof
The subspace $\mathscr{X}_{1,h}$ given by (46)–(47) guarantees that (44) holds and hence it suffices to apply the same arguments of the proof of Lemma 2.5. We omit details and refer to that proof. □

The unique solvability and convergence of (24) are provided next. We recall that the definitions of the corresponding finite element subspaces are given in (26)–(28), (30), (45), and (46).

Theorem 3.1
Let $C_0 \in]0, 1[$ and $h_0 > 0$ be the constants given in Lemmata 3.2 and 3.4, respectively. Then for all $\tilde{h} \leqslant h_0$ and for all $h \leqslant C_0\,\tilde{h}$, the mixed finite element scheme (24) has a unique solution

$(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})) \in \mathscr{X}_{1,h} \times \mathscr{M}_{1,h} \times \mathscr{M}_{h,\tilde{h}}$. In addition, there exists $C_1 > 0$, independent of h and $\tilde{h}$, such that

$$\|(\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\| \leqslant C_1 \{\|\mathscr{G}_{h,\tilde{h}}\|_{\mathscr{M}'_{h,\tilde{h}}} + \|\mathscr{N}(\mathbf{0})\|_{[L^2(\Omega)]^{2\times 2}}\}$$

where $\mathscr{G}_{h,\tilde{h}} := \mathscr{G}|_{\mathscr{M}_{h,\tilde{h}}}$ satisfies $\|\mathscr{G}_{h,\tilde{h}}\|_{\mathscr{M}'_{h,\tilde{h}}} \leqslant \|\mathbf{f}\|_{[L^2(\Omega)]^2} + \|\mathbf{g}\|_{[H^{-1/2}(\Gamma_N)]^2}$. Moreover, there exists $C_2 > 0$, independent of h and $\tilde{h}$, such that the following *Cea* estimate holds

$$\begin{aligned} &\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\| \\ &\leqslant C_2 \inf_{\substack{(\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}})) \\ \in \mathscr{X}_{1,h} \times \mathscr{M}_{1,h} \times \mathscr{M}_{h,\tilde{h}}}} \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{s}_h, (\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \boldsymbol{\delta}_h, \boldsymbol{\lambda}_{\tilde{h}}))\| \end{aligned}$$

Proof
The strong monotonicity and Lipschitz-continuity of $\mathscr{A}_1$ (cf. Lemma 2.1) are certainly valid on any subspace of $\mathscr{X}_1$. Hence, the proof follows from Lemmata 3.3 and 3.5, and a direct application of Theorems 2.2 and 2.3. □

The rate of convergence of the finite element scheme (24) is provided next.

Theorem 3.2
Assume that $\mathbf{t}|_T \in [H^1(T)]^{2\times 2}\ \forall T \in \mathscr{T}_h$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^{2\times 2}$, $\mathbf{div}\,\boldsymbol{\sigma} \in [H^1(\Omega)]^2$, $p \in H^1(\Omega)$, $\mathbf{u} \in [H^1(\Omega)]^2$, $\boldsymbol{\gamma} \in [H^1(\Omega)]^{2\times 2}$ and $\boldsymbol{\xi} \in [H^{3/2}(\Gamma_N)]^2$. Then, there exists $C > 0$, independent of h and $\tilde{h}$, such that

$$\begin{aligned} &\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\| \\ &\leqslant C\,h \left\{ \sum_{T\in\mathscr{T}_h} \|\mathbf{t}\|_{[H^1(T)]^{2\times 2}} + \|\boldsymbol{\sigma}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{[H^1(\Omega)]^2} + \|p\|_{H^1(\Omega)} \right. \\ &\qquad \left. + \|\mathbf{u}\|_{[H^1(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^1(\Omega)]^{2\times 2}} \right\} + C\tilde{h}\,\|\boldsymbol{\xi}\|_{[H^{3/2}(\Gamma_N)]^2} \end{aligned}$$

Proof
It is a straightforward consequence of the *Cea* estimate in Theorem 3.1 and the usual approximation properties of the finite element subspaces (see, e.g. References [7, 18, 19]). In particular, for $\mathscr{X}_{1,h}$ it suffices to consider the approximation property satisfied by the piecewise constant tensors. □

According to the condition $h \leqslant C_0 \tilde{h}$, we assume from now on, without loss of generality, that each edge Γ_i is contained in an edge $\tilde{\Gamma}_j$, for some $j \in \{1, \ldots, m\}$. This requires implicitly that the end points of $\tilde{\Gamma}_j$ be vertices of $\mathscr{T}_h$, which is also assumed in what follows.

4. THE *A POSTERIORI* ERROR ANALYSIS

In this section, we derive a reliable and quasi-efficient *a posteriori* error estimate for our mixed finite element scheme (24). We first let $\mathbf{X} := \mathscr{X}_1 \times \mathscr{M}_1 = [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega)$ and define the Ritz projection of the error with respect to the usual inner product $\langle\cdot,\cdot\rangle_{\mathbf{X}}$ of $\mathbf{X}$ as the unique element $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}) \in \mathbf{X}$ such that

$$\langle(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}), (\mathbf{s}, \boldsymbol{\tau}, q)\rangle_{\mathbf{X}} = [\mathscr{A}_1(\mathbf{t}), \mathbf{s}] - [\mathscr{A}_1(\mathbf{t}_h), \mathbf{s}] + [\mathscr{B}_1(\mathbf{s}), (\boldsymbol{\sigma}, p) - (\boldsymbol{\sigma}_h, p_h)] + [\mathscr{B}_1(\mathbf{t}) - \mathscr{B}_1(\mathbf{t}_h), (\boldsymbol{\tau}, q)] + [\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\xi}-\boldsymbol{\xi}_{\tilde{h}})] \qquad (48)$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q) \in \mathbf{X}$. The existence of $(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})$ is guaranteed by the Riesz representation theorem and the fact that the right-hand side of (48) constitutes a linear and bounded functional in $\mathbf{X}$ (as a mapping acting on $(\mathbf{s}, \boldsymbol{\tau}, q)$).

The following theorem provides an upper bound for $\|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|_{\mathbf{X}}$.

Theorem 4.1

Let $\bar{\mathbf{u}}_h$ be an auxiliary function in $[H^1(\Omega) \cap C(\bar{\Omega})]^2$ such that $\bar{\mathbf{u}}_h = \mathbf{0}$ on Γ_{D}. Then there exists $\bar{C} > 0$, independent of h and $\tilde{h}$, such that

$$\|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|_{\mathbf{X}}^2 \leqslant \bar{C}\{\|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla\bar{\mathbf{u}}_h\|_{[L^2(\Omega)]^{2\times 2}}^2 + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{[L^2(\Omega)]^2}^2 + \|\bar{\mathbf{u}}_h + \boldsymbol{\xi}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2}^2 + \|\boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h\mathbf{I}\|_{[L^2(\Omega)]^{2\times 2}}^2 + \|\mathrm{tr}(\mathbf{t}_h)\|_{L^2(\Omega)}^2\} \qquad (49)$$

Proof

We notice from the first two equations in (4) (equivalently (6)) that

$$[\mathscr{A}_1(\mathbf{t}), \mathbf{s}] + [\mathscr{B}_1(\mathbf{s}), (\boldsymbol{\sigma}, p)] + [\mathscr{B}_1(\mathbf{t}), (\boldsymbol{\tau}, q)] + [\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})] = 0$$

and hence (48) reduces to

$$\langle(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p}), (\mathbf{s}, \boldsymbol{\tau}, q)\rangle_{\mathbf{X}} = -[\mathscr{A}_1(\mathbf{t}_h), \mathbf{s}] - [\mathscr{B}_1(\mathbf{s}), (\boldsymbol{\sigma}_h, p_h)] - [\mathscr{B}_1(\mathbf{t}_h), (\boldsymbol{\tau}, q)] - [\mathscr{B}(\boldsymbol{\tau}, q), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}})] \quad \forall (\mathbf{s}, \boldsymbol{\tau}, q) \in \mathbf{X}$$

which, according to the definitions of $\mathscr{A}_1$, $\mathscr{B}_1$, and $\mathscr{B}$, is equivalent to

$$\langle\bar{\mathbf{t}}, \mathbf{s}\rangle_{[L^2(\Omega)]^{2\times 2}} = \int_\Omega (\boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h\mathbf{I}) : \mathbf{s} \quad \forall \mathbf{s} \in [L^2(\Omega)]^{2\times 2} \qquad (50)$$

$$\langle\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau}\rangle_{H(\mathbf{div};\Omega)} = \int_\Omega (\mathbf{t}_h + \boldsymbol{\gamma}_h) : \boldsymbol{\tau} + \int_\Omega \mathbf{u}_h \cdot \mathbf{div}\,\boldsymbol{\tau} + \langle\boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{\xi}_{\tilde{h}}\rangle_{\Gamma_{\mathrm{N}}} \quad \forall \boldsymbol{\tau} \in H(\mathbf{div};\Omega) \qquad (51)$$

and

$$\langle \bar{p}, q \rangle_{L^2(\Omega)} = - \int_\Omega q \, \mathrm{tr}(\mathbf{t}_h) \quad \forall q \in L^2(\Omega) \tag{52}$$

It follows straightforward from (50) and (52) that

$$\bar{\mathbf{t}} = \boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h \mathbf{I} \quad \text{and} \quad \bar{p} = - \mathrm{tr}(\mathbf{t}_h) \tag{53}$$

On the other hand, substracting and adding $\bar{\mathbf{u}}_h$ in the second term on the right-hand side of (51), and then integrating by parts on Ω, we find that

$$\langle \bar{\boldsymbol{\sigma}}, \boldsymbol{\tau} \rangle_{H(\mathbf{div};\Omega)} = \int_\Omega (\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \bar{\mathbf{u}}_h) : \boldsymbol{\tau} + \int_\Omega (\mathbf{u}_h - \bar{\mathbf{u}}_h) \cdot \mathbf{div}\, \boldsymbol{\tau} + \langle \boldsymbol{\tau}\boldsymbol{\nu}, \bar{\mathbf{u}}_h + \boldsymbol{\xi}_{\tilde{h}} \rangle_{\Gamma_{\mathrm{N}}} \quad \forall \boldsymbol{\tau} \in H(\mathbf{div}; \Omega)$$

which yields

$$\|\bar{\boldsymbol{\sigma}}\|^2_{H(\mathbf{div};\Omega)} \leqslant C \, \{ \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \bar{\mathbf{u}}_h\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{[L^2(\Omega)]^2} + \|\bar{\mathbf{u}}_h + \boldsymbol{\xi}_{\tilde{h}}\|^2_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2} \} \tag{54}$$

In this way, (53) and (54) imply (49) and complete the proof. □

In order to continue the analysis we need to introduce further notations. Given $T \in \mathscr{T}_h$, we denote by $E(T)$ the set of its edges, and by E_h the set of all edges of the triangulation $\mathscr{T}_h$. Then we can write $E_h = E_h(\Omega) \cup E_h(\Gamma_{\mathrm{N}}) \cup E_h(\Gamma_{\mathrm{D}})$, where $E_h(\Omega) := \{e \in E_h \, : \, e \subseteq \Omega\}$, $E_h(\Gamma_{\mathrm{N}}) := \{e \in E_h \, : \, e \subseteq \Gamma_{\mathrm{N}}\}$, and similarly for $E_h(\Gamma_{\mathrm{D}})$. In what follows, h_e stands for the length of edge $e \in E_h$, and for each $e \subseteq E_h(\Gamma_{\mathrm{N}})$ we set $\tilde{h}_e := |\tilde{\Gamma}_j|$, where $\tilde{\Gamma}_j$ is the segment containing edge e. In addition, the tangential vector on Γ_{N} is given by $s := (-v_2, v_1)^{\mathtt{t}}$ where $(v_1, v_2)^{\mathtt{t}}$ is the corresponding unit outward normal.

We now establish a reliable *a posteriori* error estimate for our mixed finite element scheme (24).

Theorem 4.2
Let $\bar{\mathbf{u}}_h$ be an auxiliary function in $[H^1(\Omega) \cap C(\bar{\Omega})]^2$ such that $\bar{\mathbf{u}}_h = \mathbf{0}$ on Γ_{D} and $\bar{\mathbf{u}}_h(\mathbf{x}) = -\boldsymbol{\xi}_{\tilde{h}}(\mathbf{x})$ on each vertex $\mathbf{x}$ of $\mathscr{T}_h$ lying on Γ_{N}. Assume that $\mathscr{A}_1$ has a continuous first-order Gâteaux derivative $\mathscr{D}\mathscr{A}_1$ and that the Neumann data $\mathbf{g} \in [L^2(\Gamma_{\mathrm{N}})]^2$. Then there exists $C_{\mathtt{rel}} > 0$, independent of h and $\tilde{h}$, such that

$$\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|^2 \leqslant C_{\mathtt{rel}} \, \boldsymbol{\theta}^2 \tag{55}$$

where $\boldsymbol{\theta}^2 := \sum_{T \in \mathscr{T}_h} \theta_T^2$, and for each $T \in \mathscr{T}_h$,

$$\theta_T^2 := \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla \bar{\mathbf{u}}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h \mathbf{I}\|^2_{[L^2(T)]^{2\times 2}}$$
$$+ \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)} + \log[1 + C_h(\Gamma_{\mathrm{N}})] \sum_{e \in E(T) \cap E_h(\Gamma_{\mathrm{N}})} h_e \left\| \frac{\mathrm{d}\bar{\mathbf{u}}_h}{\mathrm{d}s} + \frac{\mathrm{d}\boldsymbol{\xi}_{\tilde{h}}}{\mathrm{d}s} \right\|^2_{[L^2(e)]^2}$$

$$+\|\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2}+\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathrm{t}}\|^2_{[L^2(T)]^{2\times 2}}+\log[1+C_{\tilde{h}}(\Gamma_{\mathrm{N}})]$$
$$\times\sum_{e\in E(T)\cap E_h(\Gamma_{\mathrm{N}})}\tilde{h}_e\|\boldsymbol{\sigma}_h\boldsymbol{\nu}-\mathbf{g}\|^2_{[L^2(e)]^2}$$

with $C_h(\Gamma_{\mathrm{N}}):=\max\{|\Gamma_i|/|\Gamma_j|:|i-j|=1\}$ and $C_{\tilde{h}}(\Gamma_{\mathrm{N}}):=\max\{|\tilde{\Gamma}_i|/|\tilde{\Gamma}_j|:|i-j|=1\}$.

Proof
Since $\mathscr{A}_1$ is strongly monotone and Lipschitz continuous (cf. Lemma 2.1), we find that the Gâteaux derivative $\mathscr{D}\mathscr{A}_1(\mathbf{r})(\cdot,\cdot)$ becomes a uniformly bounded and elliptic bilinear form on $\mathscr{X}_1\times\mathscr{X}_1$ for all $\mathbf{r}\in\mathscr{X}_1$. In addition, the continuity of $\mathscr{D}\mathscr{A}_1$ guarantees that there exists $\bar{\mathbf{r}}\in\mathscr{X}_1$ such that

$$\mathscr{D}\mathscr{A}_1(\bar{\mathbf{r}})(\mathbf{t}-\mathbf{t}_h,\mathbf{s})=[\mathscr{A}_1(\mathbf{t})-\mathscr{A}_1(\mathbf{t}_h),\mathbf{s}]\quad\forall\mathbf{s}\in\mathscr{X}_1 \tag{56}$$

Hence, replacing $[\mathscr{A}_1(\cdot),\mathbf{s}]:=\int_\Omega\mathscr{N}(\cdot):\mathbf{s}$ in (4) by $\mathscr{D}\mathscr{A}_1(\bar{\mathbf{r}})(\cdot,\mathbf{s})$, and then adding the three equations on the left-hand side of (4), we obtain a linear operator for which (23) (cf. Theorem 2.4) constitutes a global inf–sup condition. This means that there exists $C>0$, independent of h and $\tilde{h}$, such that for each $(\vec{\mathbf{t}},(\vec{\boldsymbol{\sigma}},\vec{p}),(\vec{\mathbf{u}},\vec{\boldsymbol{\gamma}},\vec{\boldsymbol{\xi}}))\in\mathscr{X}_1\times\mathscr{M}_1\times\mathscr{M}$ there holds

$$C\|(\vec{\mathbf{t}},(\vec{\boldsymbol{\sigma}},\vec{p}),(\vec{\mathbf{u}},\vec{\boldsymbol{\gamma}},\vec{\boldsymbol{\xi}}))\|\leqslant\sup_{\substack{(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\in\mathscr{X}_1\times\mathscr{M}_1\times\mathscr{M}\\ \|(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\|\leqslant 1}}\{\mathscr{D}\mathscr{A}_1(\bar{\mathbf{r}})(\vec{\mathbf{t}},\mathbf{s})+[\mathscr{B}_1(\vec{\mathbf{t}}),(\boldsymbol{\tau},q)]$$
$$+[\mathscr{B}_1(\mathbf{s}),(\vec{\boldsymbol{\sigma}},\vec{p})]+[\mathscr{B}(\boldsymbol{\tau},q),(\vec{\mathbf{u}},\vec{\boldsymbol{\gamma}},\vec{\boldsymbol{\xi}})]+[\mathscr{B}(\vec{\boldsymbol{\sigma}},\vec{p}),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]\} \tag{57}$$

Thus, applying estimate (57) to the Galerkin error, that is taking

$$(\vec{\mathbf{t}},(\vec{\boldsymbol{\sigma}},\vec{p}),(\vec{\mathbf{u}},\vec{\boldsymbol{\gamma}},\vec{\boldsymbol{\xi}})):=(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}))-(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))$$

and using (48) and (56), we find that

$$C\|(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}))-(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))\|$$
$$\leqslant\sup_{\substack{(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\in\mathscr{X}_1\times\mathscr{M}_1\times\mathscr{M}\\ \|(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\|\leqslant 1}}\{\langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_{\mathbf{X}}+[\mathscr{B}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,p-p_h),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda})]\}$$

which, according to the definition of $\mathscr{B}$ and the third equation of (4), yields

$$C\|(\mathbf{t},(\boldsymbol{\sigma},p),(\mathbf{u},\boldsymbol{\gamma},\boldsymbol{\xi}))-(\mathbf{t}_h,(\boldsymbol{\sigma}_h,p_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h,\boldsymbol{\xi}_{\tilde{h}}))\|$$
$$\leqslant\sup_{\substack{(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\in\mathscr{X}_1\times\mathscr{M}_1\times\mathscr{M}\\ \|(\mathbf{s},(\boldsymbol{\tau},q),(\mathbf{v},\boldsymbol{\delta},\boldsymbol{\lambda}))\|\leqslant 1}}\left\{\langle(\bar{\mathbf{t}},\bar{\boldsymbol{\sigma}},\bar{p}),(\mathbf{s},\boldsymbol{\tau},q)\rangle_{\mathbf{X}}+\int_\Omega(\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}_h)\cdot\mathbf{v}\right.$$
$$\left.+\int_\Omega\boldsymbol{\sigma}_h:\boldsymbol{\delta}+\langle\boldsymbol{\sigma}_h\boldsymbol{\nu}-\mathbf{g},\boldsymbol{\lambda}\rangle_{\Gamma_{\mathrm{N}}}\right\}$$

Next, noting that $\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\delta} = \frac{1}{2}\int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}) : \boldsymbol{\delta}$, and applying Cauchy–Schwarz's inequality and the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma_{\mathrm{N}}}$, we deduce that there exists $\tilde{C} > 0$, independent of h and $\tilde{h}$, such that

$$\begin{aligned}
&\tilde{C}\,\|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \gamma, \xi)) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \gamma_h, \xi_{\tilde{h}}))\|^2 \\
&\quad \leqslant \|(\bar{\mathbf{t}}, \bar{\boldsymbol{\sigma}}, \bar{p})\|^2_{\mathbf{X}} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}
\end{aligned} \tag{58}$$

We now provide a suitable upper bound for $\|\bar{\mathbf{u}}_h + \xi_{\tilde{h}}\|^2_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2}$ (see right-hand side of (49)) in terms of L^2-local norms on the edges of Γ_{N} (see References [25, 26]). Indeed, since $\bar{\mathbf{u}}_h + \xi_{\tilde{h}}$ vanishes at the nodes of Γ_{N}, we can apply Theorem 1 in Reference [25] to deduce that there exists $C > 0$, independent of h and $\tilde{h}$, such that

$$\|\bar{\mathbf{u}}_h + \xi_{\tilde{h}}\|^2_{[H^{1/2}_{00}(\Gamma_{\mathrm{N}})]^2} \leqslant C\, \log[1 + C_h(\Gamma_{\mathrm{N}})] \sum_{e \in E_h(\Gamma_{\mathrm{N}})} h_e \left\| \frac{\mathrm{d}\bar{\mathbf{u}}_h}{\mathrm{d}s} + \frac{\mathrm{d}\xi_{\tilde{h}}}{\mathrm{d}s} \right\|^2_{[L^2(e)]^2} \tag{59}$$

Similarly, we need to bound the Neumann residual $\|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2}$ by local computable quantities on the segments $\tilde{\Gamma}_j$, $j \in \{1, \ldots, m\}$. Because of the definition of the subspace $\mathcal{M}^{\boldsymbol{\sigma}}_{1,h}$, it is easy to see that $\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g} \in [L^2(\Gamma_{\mathrm{N}})]^2$. In addition, taking $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\delta} = \mathbf{0}$ in the third equation of (24), we find that $\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}$ is $[L^2(\Gamma_{\mathrm{N}})]^2$-orthogonal to the finite element subspace $\mathcal{M}^{\xi}_{\tilde{h}}$, and then, a straightforward application of Theorem 2 in Reference [25] yields

$$\|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2} \leqslant \log\,[1 + C_{\tilde{h}}(\Gamma_{\mathrm{N}})] \sum_{j=1}^{m} |\tilde{\Gamma}_j|\, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(\tilde{\Gamma}_j)]^2}$$

Since each edge $e \in E_h(\Gamma_{\mathrm{N}})$ is contained in a segment $\tilde{\Gamma}_j$, for some $j \in \{1, \ldots, m\}$, we find that $\sum_{j=1}^{m} |\tilde{\Gamma}_j|\, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(\tilde{\Gamma}_j)]^2} = \sum_{e \in E_h(\Gamma_{\mathrm{N}})} \tilde{h}_e\, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2}$, whence the above inequality becomes

$$\|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[H^{-1/2}(\Gamma_{\mathrm{N}})]^2} \leqslant \log[1 + C_{\tilde{h}}(\Gamma_{\mathrm{N}})] \sum_{e \in E_h(\Gamma_{\mathrm{N}})} \tilde{h}_e\, \|\boldsymbol{\sigma}_h \boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2} \tag{60}$$

Finally, the reliability estimate (55) follows directly from (58), (49), (59), and (60). □

It is interesting to observe here that all the terms defining θ_T, but the first, the second, and the fifth (which depend on the auxiliary function), are residual expressions corresponding to the nonlinear constitutive equation, the incompressibility condition, the equilibrium equation, the symmetry of $\boldsymbol{\sigma}$, and the Neumann boundary condition, respectively. Nevertheless, we emphasize that, because of those three terms depending on $\bar{\mathbf{u}}_h$, the performance of an adaptive algorithm based on $\boldsymbol{\theta}$ will also be determined by this auxiliary function. In other words, the eventual efficiency, the rate of convergence, and the capability of the method to localize the singularities of the boundary value problem, will certainly depend on the residual terms as well as on the choice of $\bar{\mathbf{u}}_h$. We will come back to this issue at the end of the present section.

We now aim to prove that $\boldsymbol{\theta}$ is *quasi-efficient*, which will make use of Lemmata 4.1 and 4.2 below. The proofs of these results employ inverse inequalities in finite element subspaces and the localization technique based on triangle-bubble and edge-bubble functions (see References [27–29]).

Lemma 4.1
There exists $C>0$, independent of h and $\tilde{h}$, such that

$$\sum_{e\in E_h(\Gamma_N)} h_e \left\| \frac{d\bar{\mathbf{u}}_h}{ds} + \frac{d\boldsymbol{\xi}_{\tilde{h}}}{ds} \right\|^2_{[L^2(e)]^2} \leqslant C\{\|\boldsymbol{\xi} - \boldsymbol{\xi}_{\tilde{h}}\|^2_{[H^{1/2}_{00}(\Gamma_N)]^2} + \|\mathbf{u} - \bar{\mathbf{u}}_h\|^2_{[H^1(\Omega)]^2}\}$$

Proof
It is a componentwise application of Lemma 4.5 in Reference [30] (see also Reference [29, Lemma 5.7]). □

Lemma 4.2
There exists $c>0$, independent of h and $\tilde{h}$, such that

$$\sum_{e\in E_h(\Gamma_N)} \tilde{h}_e \|\boldsymbol{\sigma}_h \mathbf{v} - \mathbf{g}\|^2_{[L^2(e)]^2} \leqslant c\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{H(\mathbf{div};\Omega)}$$

Proof
It is an adaptation of the proof of Lemma 6.5 in Reference [27] (see also Reference [29, Lemma 5.9 and Equation (5.25)]). □

The following theorem establishes the *quasi-efficiency* of the *a posteriori* error estimate $\boldsymbol{\theta}$. This modified concept refers to the extra term appearing below on the right-hand side of (61).

Theorem 4.3
Assume that $\mathbf{u}\in[H^1(\Omega)]^2$. Then there exists $C_{\mathtt{eff}}>0$, independent of h and $\tilde{h}$, such that

$$C_{\mathtt{eff}}\boldsymbol{\theta}^2 \leqslant \|(\mathbf{t}, (\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}, \boldsymbol{\xi})) - (\mathbf{t}_h, (\boldsymbol{\sigma}_h, p_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h, \boldsymbol{\xi}_{\tilde{h}}))\|^2 + \|\mathbf{u} - \bar{\mathbf{u}}_h\|^2_{[H^1(\Omega)]^2} \tag{61}$$

Proof
We first observe that

$$\|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)} = \|\mathrm{tr}(\mathbf{t} - \mathbf{t}_h)\|^2_{L^2(T)} \leqslant 2\,\|\mathbf{t} - \mathbf{t}_h\|^2_{[L^2(T)]^{2\times 2}} \tag{62}$$

and

$$\|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} = \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} \tag{63}$$

Next, adding and substracting the exact unknowns in the corresponding terms defining θ_T^2, using that $\nabla\mathbf{u} = \mathbf{t} + \boldsymbol{\gamma}$, that $\boldsymbol{\sigma} = \mathscr{N}(\mathbf{t}) + p\,\mathbf{I}$, that $\mathscr{N}$ is Lipschitz-continuous (cf. (2)), and that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}}$, we obtain that

$$\begin{aligned} &\|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla\bar{\mathbf{u}}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{[L^2(T)]^2} \\ &\quad \leqslant C\{\|\mathbf{t}-\mathbf{t}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}-\mathbf{u}_h\|^2_{[L^2(T)]^2} + \|\mathbf{u}-\bar{\mathbf{u}}_h\|^2_{[H^1(T)]^2}\} \end{aligned} \tag{64}$$

and

$$\|\boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathrm{t}}\|^2_{[L^2(T)]^{2\times 2}}$$
$$\leqslant C\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{t} - \mathbf{t}_h\|^2_{[L^2(T)]^{2\times 2}} + \|p - p_h\|^2_{L^2(T)}\} \tag{65}$$

where $C>0$ is independent of h and $\tilde{h}$.

Finally, (61) follows easily from (62)–(65), Lemmata 4.1 and 4.2, and the quasi-uniformity of the meshes on Γ_{N}. □

The *quasi-efficiency* estimate (61) suggests to choose $\bar{\mathbf{u}}_h$ as a higher order approximation of the exact solution $\mathbf{u}$, thus yielding full efficiency of $\boldsymbol{\theta}$. However, since achieving this may require a too expensive procedure, we just proceed in an heuristic sense and propose the following postprocessing to define $\bar{\mathbf{u}}_h$. We first compute local functions $\bar{\mathbf{u}}_{h,T}$ for each $T\in\mathscr{T}_h$, satisfying the following conditions:

1. $\bar{\mathbf{u}}_{h,T}\in[\tilde{\mathbf{P}}_2(T)]^2$, where $\tilde{\mathbf{P}}_2(T)=\mathrm{span}\{1, x_1, x_2, x_1^2, x_2^2\}$.
2. $\nabla\bar{\mathbf{u}}_{h,T}$ is the $[L^2(T)]^{2\times 2}$-projection of $(\mathbf{t}_h+\boldsymbol{\gamma}_h)|_T$ onto the space $[\mathbf{P}_1(T)]^{2\times 2}$.
3. $\bar{\mathbf{u}}_{h,T}(\mathbf{x}_T)=\mathbf{u}_h|_T$, where $\mathbf{x}_T$ is the barycentre of the triangle T.

We remark that each $\bar{\mathbf{u}}_{h,T}$ is uniquely determined by these constraints. Then, we define $\bar{\mathbf{u}}_h$ as the unique function in $[C(\bar{\Omega})]^2$ such that:

1. $\bar{\mathbf{u}}_h|_T\in[\mathbf{P}_2(T)]^2$ for each $T\in\mathscr{T}_h$.
2. For each vertex $\mathbf{x}$ of $\mathscr{T}_h$ lying on Γ_{N} and for each middle point $\mathbf{x}$ of the edges $e\in E_h(\Gamma_{\mathrm{N}})$, $\bar{\mathbf{u}}_h(\mathbf{x}) = -\boldsymbol{\xi}_{\tilde{h}}(\mathbf{x})$.
3. For each vertex $\mathbf{x}$ of $\mathscr{T}_h$ lying on Γ_{D} and for each middle point $\mathbf{x}$ of the edges $e\in E_h(\Gamma_{\mathrm{D}})$, $\bar{\mathbf{u}}_h(\mathbf{x})=\mathbf{0}$.
4. For each vertex $\mathbf{x}$ of $\mathscr{T}_h$ lying in Ω and for each middle point $\mathbf{x}$ of the edges $e\in E_h(\Omega)$, $\bar{\mathbf{u}}_h(\mathbf{x})$ is the average of the values $\bar{\mathbf{u}}_{h,T}(\mathbf{x})$ on all the triangles $T\in\mathscr{T}_h$ to which $\mathbf{x}$ belongs.

It is easy to see that the present choice of $\bar{\mathbf{u}}_h$ guarantees that $\bar{\mathbf{u}}_h=\mathbf{0}$ on Γ_{D}. Furthermore, since $\bar{\mathbf{u}}_h$ is the piecewise quadratic interpolant of the piecewise linear function $-\boldsymbol{\xi}_{\tilde{h}}$ on Γ_{N}, we find that $\bar{\mathbf{u}}_h+\boldsymbol{\xi}_{\tilde{h}}$ vanishes identically on the whole Neumann boundary, and hence θ_T^2 reduces to

$$\theta_T^2 := \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla\bar{\mathbf{u}}_h\|^2_{[L^2(T)]^{2\times 2}} + \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h - \mathscr{N}(\mathbf{t}_h) - p_h\,\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}}$$
$$+ \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)} + \|\mathbf{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathrm{t}}\|^2_{[L^2(T)]^{2\times 2}}$$
$$+ \log[1 + C_{\tilde{h}}(\Gamma_{\mathrm{N}})] \sum_{e\in E(T)\cap E_h(\Gamma_{\mathrm{N}})} \tilde{h}_e\,\|\boldsymbol{\sigma}_h\boldsymbol{\nu} - \mathbf{g}\|^2_{[L^2(e)]^2} \tag{66}$$

5. NUMERICAL RESULTS

In this section, we present some numerical results illustrating the performance of the mixed finite element scheme (24) and the *a posteriori* error estimate $\boldsymbol{\theta}$, with the above described choice of $\bar{\mathbf{u}}_h$. We begin by introducing additional notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathscr{X}_{1,h}$, $\mathscr{M}_{1,h}$, and $\mathscr{M}_{h,\tilde{h}}$. Also, the individual and total errors are denoted by

$$\mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{[L^2(\Omega)]^{2\times 2}}, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}, \quad \mathsf{e}(p) := \|p - p_h\|_{L^2(\Omega)}$$

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{[L^2(\Omega)]^2}, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(\Omega)]^{2\times 2}}, \quad \mathsf{e}(\boldsymbol{\xi}) := \|\boldsymbol{\xi} - \boldsymbol{\xi}_{\tilde{h}}\|_{[H_{00}^{1/2}(\Gamma_{\mathrm{N}})]^2}$$

and

$$\mathsf{e} := \{[\mathsf{e}(\mathbf{t})]^2 + [\mathsf{e}(\boldsymbol{\sigma})]^2 + [\mathsf{e}(p)]^2 + [\mathsf{e}(\mathbf{u})]^2 + [\mathsf{e}(\boldsymbol{\gamma})]^2 + [\mathsf{e}(\boldsymbol{\xi})]^2\}^{1/2}$$

Then, given two consecutive triangulations with degrees of freedom N and N' and corresponding total errors e and e', respectively, we define the experimental rate of convergence by

$$r(\mathsf{e}) := -2\,\frac{\log(\mathsf{e}/\mathsf{e}')}{\log(N/N')}$$

The adaptative algorithm to be used in the computation of the solutions of (24) reads (see Reference [31]):

1. Start with a coarse mesh $\mathscr{T}_h$.
2. Solve the Galerkin scheme (24) for the current mesh $\mathscr{T}_h$.
3. Compute θ_T for each triangle $T \in \mathscr{T}_h$.
4. Consider stopping criterion and decide to finish or go to next step.
5. Use *blue–green* procedure (see Reference [31]) to refine each element $T' \in \mathscr{T}_h$ whose local indicator $\theta_{T'}$ satisfies $\theta_{T'} \geqslant \frac{1}{2}\max\{\theta_T : T \in \mathscr{T}_h\}$.
6. Define resulting mesh as the new $\mathscr{T}_h$ and go to step 2.

We now specify the two examples to be considered. The nonlinear operator $\mathscr{N}$ is given in both cases by

$$\mathscr{N}(\mathbf{r}) := \left(\frac{3}{2} - \mu(\|\mathbf{r}^{\mathrm{d}}\|^2)\right)\operatorname{tr}(\mathbf{r})\,\mathbf{I} + 2\mu(\|\mathbf{r}^{\mathrm{d}}\|^2)\mathbf{r} \quad \forall \mathbf{r} \in [L^2(\Omega)]^{2\times 2}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}$ is the nonlinear Lamé function defined by $\mu(t) = 1 + (1/2(1+t))$ for all $t \in \mathbb{R}^+$. This corresponds to a hyperelastic material whose constitutive equation is given by the Hencky–von Mises stress–strain relation (see References [1, 2, 4]). Since $\mu \in C^1(\mathbb{R}^+)$, we find that the nonlinear operator $\mathscr{A}_1$ has a continuous first-order Gâteaux derivative $\mathscr{D}\mathscr{A}_1$. In addition, it is easy to see that there exists constants μ_1, μ_2 such that $0 < \mu_1 \leqslant \mu(t) < \frac{3}{2}$ and $0 < \mu_1 \leqslant \mu(t) + 2t\mu'(t) \leqslant \mu_2$ for all $t \in \mathbb{R}^+$, which allows to show (see, e.g. Reference [1, Lemma 5.1]) that $\mathscr{A}_1$ becomes strongly monotone and Lipschitz continuous.

In Example 1, we consider a benchmark problem taken from Chapter 4 in Reference [31], where Ω is given by the L-shaped domain $]-1,1[^2 - [0,1] \times [-1,0]$, $\Gamma_{\mathrm{D}} := [0,1] \times \{0\} \cup \{0\}$,

$\times[-1,0]$ and $\Gamma_N := \Gamma - \Gamma_D$. The data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that $(\mathbf{u}, p)$ is given in polar coordinates by

$$\mathbf{u}(r,\theta) := r^{\alpha}\{(1+\alpha)z(\theta)(\sin(\theta), -\cos(\theta)) + z'(\theta)(\cos(\theta), \sin(\theta))\}$$

$$p(r,\theta) := \frac{r^{\alpha-1}}{(1-\alpha)}\{(1+\alpha)^2 z'(\theta) + z'''(\theta)\}$$

where $\alpha = 856\,399/1\,572\,864 \approx 0.54448$, and the function z is defined, with $\omega = 3\pi/2$, as follows:

$$z(\theta) := \frac{\sin((1+\alpha)\theta)\cos(\alpha\omega)}{(1+\alpha)} - \cos((1+\alpha)\theta) - \frac{\sin((1-\alpha)\theta)\cos(\alpha\omega)}{(1-\alpha)} + \cos((1-\alpha)\theta)$$

It is not difficult to see that $\mathbf{u}$ is divergence free in Ω and that p and the partial derivatives of $\mathbf{u}$ have a singular behaviour at the re-entrant corner $(0,0) \in \Omega$. Also, we note that $\mathbf{u}$ vanishes at Γ_D, which holds for $\theta = 0$ and $3\pi/2$.

Next, in Example 2, we take $\Omega :=]0,2[^2 - \bar{\mathbf{B}}(\mathbf{0},1)$, where $\bar{\mathbf{B}}(\mathbf{0},1)$ is the closed unit ball in $\mathbb{R}^2$, $\Gamma_D := \{\mathbf{x} := (x_1,x_2)^t \in \bar{\Omega} : x_1^2 + x_2^2 = 1\}$, and $\Gamma_N := \Gamma - \Gamma_D$. The data $\mathbf{f}$ and $\mathbf{g}$ are chosen so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) := 5(1 - x_1^2 - x_2^2)\mathrm{e}^{-5\,(1-x_1^2-x_2^2)^2}(x_1, -x_2)$$

$$p(\mathbf{x}) := \sin(x_1\,x_2)$$

for all $\mathbf{x} := (x_1,x_2)^t \in \Omega$. We observe here that $\mathbf{u}$ vanishes at Γ_D, is divergence free in Ω, and has an inner layer around the unit circle.

The numerical results shown below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a MATLAB code. We remark that the mixed finite element scheme (24) becomes a nonlinear algebraic system with N unknowns, which is solved by Newton's method with tolerance 10^{-3} for the relative error. Also, according to the requirement established in Theorem 3.1 for the mesh sizes h and $\tilde{h}$, and since the constant C_0 mentioned there is not explicitly known, we simply set a vertex of the independent partition $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_m\}$ every two vertices of $\mathcal{T}_h$ on Γ_N. As we will see below, this choice works out well in both examples. In addition, there is no need of taking sufficiently small values of $\tilde{h}$ (as technically suggested by the inequality $\tilde{h} \leqslant h_0$ in Theorem 3.1) since the resulting discrete schemes are all well posed.

In Tables I–IV, we give the individual errors (computed on each triangle using a Gaussian quadrature rule), the effectivity index $\boldsymbol{\theta}/\mathrm{e}$, and the corresponding experimental rates of convergence for the uniform and adaptive refinements. We observe there that the errors of the adaptive procedure decrease much faster than those obtained by the uniform one. Furthermore, the effectivity indexes remain all bounded above and below, which confirms the reliability of $\boldsymbol{\theta}$ and provides numerical evidences for its eventual efficiency. It is also interesting to notice that the dominant component of the total error e is given by $\mathrm{e}(\boldsymbol{\sigma})$, which is particularly notorious in Example 2. Next, in Figures 1 and 2 we display the total error e vs the degrees of freedom

Table I. Individual errors, effectivity index, and rate of convergence for the uniform refinement (Example 1).

N	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\xi})$	$\boldsymbol{\theta}/e$	$r(e)$
431	1.6827	4.3206	1.8347	1.7801	2.5933	3.0995	0.6791	—
1655	1.0580	2.8794	1.2782	0.7783	1.2534	1.2072	0.6718	0.8221
6479	0.7035	1.9895	0.8866	0.3350	0.6894	0.4757	0.6894	0.6491
25 631	0.4791	1.3994	0.6169	0.1518	0.4293	0.2036	0.7033	0.5561
101 951	0.3280	0.9885	0.4265	0.0710	0.2823	0.0915	0.7170	0.5276

Table II. Individual errors, effectivity index, and rate of convergence for the adaptive refinement (Example 1).

N	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\xi})$	$\boldsymbol{\theta}/e$	$r(e)$
431	1.6827	4.3206	1.8347	1.7801	2.5933	3.0995	0.6790	—
896	1.2404	3.2467	1.3897	1.1565	1.5600	2.0345	0.6790	0.9634
1701	0.9301	2.6065	1.1770	0.7441	1.0234	1.2455	0.7749	0.9146
2981	0.7296	2.0713	0.9130	0.4964	0.6820	0.6414	0.6581	1.0468
3314	0.6772	1.9334	0.8484	0.4688	0.6063	0.5781	0.6491	1.4107
4043	0.5961	1.6780	0.7260	0.4167	0.4984	0.5330	0.6465	1.4175
4815	0.5624	1.5887	0.6974	0.3704	0.4175	0.4211	0.6886	0.8211
8679	0.4657	1.3089	0.5710	0.2701	0.3094	0.2280	0.6426	0.7335
12 382	0.3793	1.0489	0.4549	0.2076	0.2490	0.1894	0.6408	1.2436
13 642	0.3614	0.9973	0.4322	0.1994	0.2211	0.1559	0.6278	1.1433
16 050	0.3368	0.9212	0.3975	0.1885	0.1970	0.1405	0.6354	0.9856
28 569	0.2493	0.6862	0.2902	0.1425	0.1343	0.1009	0.6568	1.0419
38 175	0.2201	0.5958	0.2529	0.1217	0.1119	0.0622	0.6476	0.9971
46 919	0.1978	0.5363	0.2261	0.1058	0.1009	0.0448	0.6424	1.0535

Table III. Individual errors, effectivity index, and rate of convergence for the uniform refinement (Example 2).

N	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	$e(\boldsymbol{\xi})$	$\boldsymbol{\theta}/e$	$r(e)$
220	4.2863	52.7757	1.6463	0.9703	4.0542	2.3792	0.9882	—
837	4.1703	46.8548	4.3708	1.8337	5.6152	3.6741	0.9974	0.1614
3259	2.4477	34.0130	1.4432	0.9578	2.9138	1.9283	1.0008	0.4858
12 855	1.0550	18.2077	0.4385	0.1136	0.5552	0.1741	0.9996	0.9203
51 055	0.5236	10.4560	0.2069	0.0536	0.1448	0.0387	0.9993	0.8056
203 348	0.2611	5.9983	0.1015	0.0261	0.0628	0.0096	0.9993	0.8043

N for both refinements. The faster decreasing of e observed in these figures for the adaptive algorithm is certainly in agreement with the individual errors and the experimental rates of convergence provided in the tables. As shown by the values of $r(e)$ in each example, the adaptive method is able to recover the quasi-optimal rate of convergence $O(h)$ for the global error e.

Table IV. Individual errors, effectivity index, and rate of convergence for the adaptive refinement (Example 2).

N	$\mathtt{e}(\mathbf{t})$	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\gamma})$	$\mathtt{e}(\boldsymbol{\xi})$	$\boldsymbol{\theta}/\mathtt{e}$	$r(\mathtt{e})$
220	4.2863	52.7757	1.6463	0.9703	4.0542	2.3792	0.9882	—
626	4.2099	46.7481	3.7205	1.1941	4.3224	2.6518	1.0070	0.2216
1328	2.7281	34.2342	1.7934	1.3762	3.4172	3.1041	1.0089	0.8258
3482	1.3944	19.0088	1.2002	0.3696	0.8691	0.9548	1.0199	1.2354
10 704	0.6876	11.5406	0.6672	0.1657	0.4562	0.4021	1.0032	0.8926
19 068	0.5262	9.3770	0.5811	0.1683	0.4357	0.3671	0.9995	0.7172
46 203	0.3399	5.8507	0.3321	0.0873	0.2519	0.1525	0.9964	1.0677
67 564	0.2850	4.9443	0.2790	0.0717	0.2003	0.1266	0.9968	0.8866
150 350	0.1999	3.2427	0.2102	0.0316	0.1297	0.0564	0.9956	1.0535
210 532	0.1644	2.7240	0.1643	0.0286	0.1036	0.0461	0.9945	1.0380

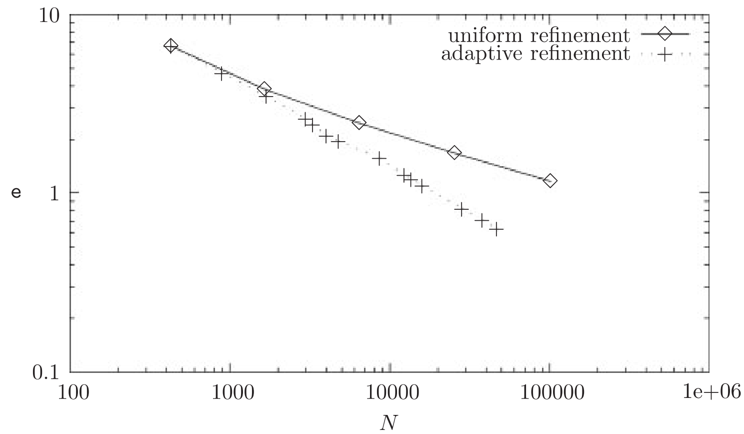

Figure 1. Total error $\mathtt{e}$ vs degrees of freedom N for both refinements (Example 1).

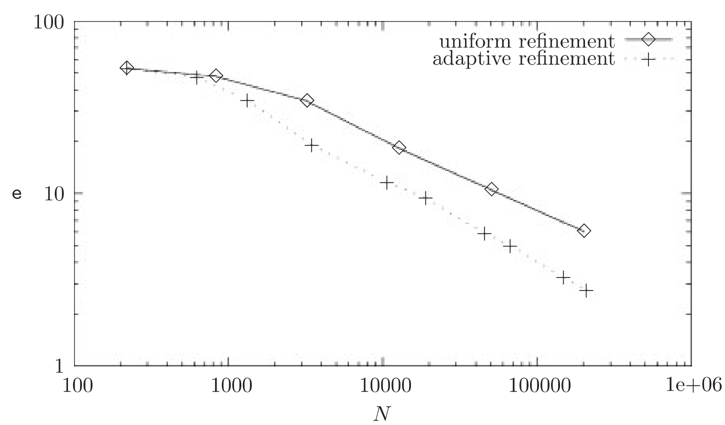

Figure 2. Total error $\mathtt{e}$ vs degrees of freedom N for both refinements (Example 2).

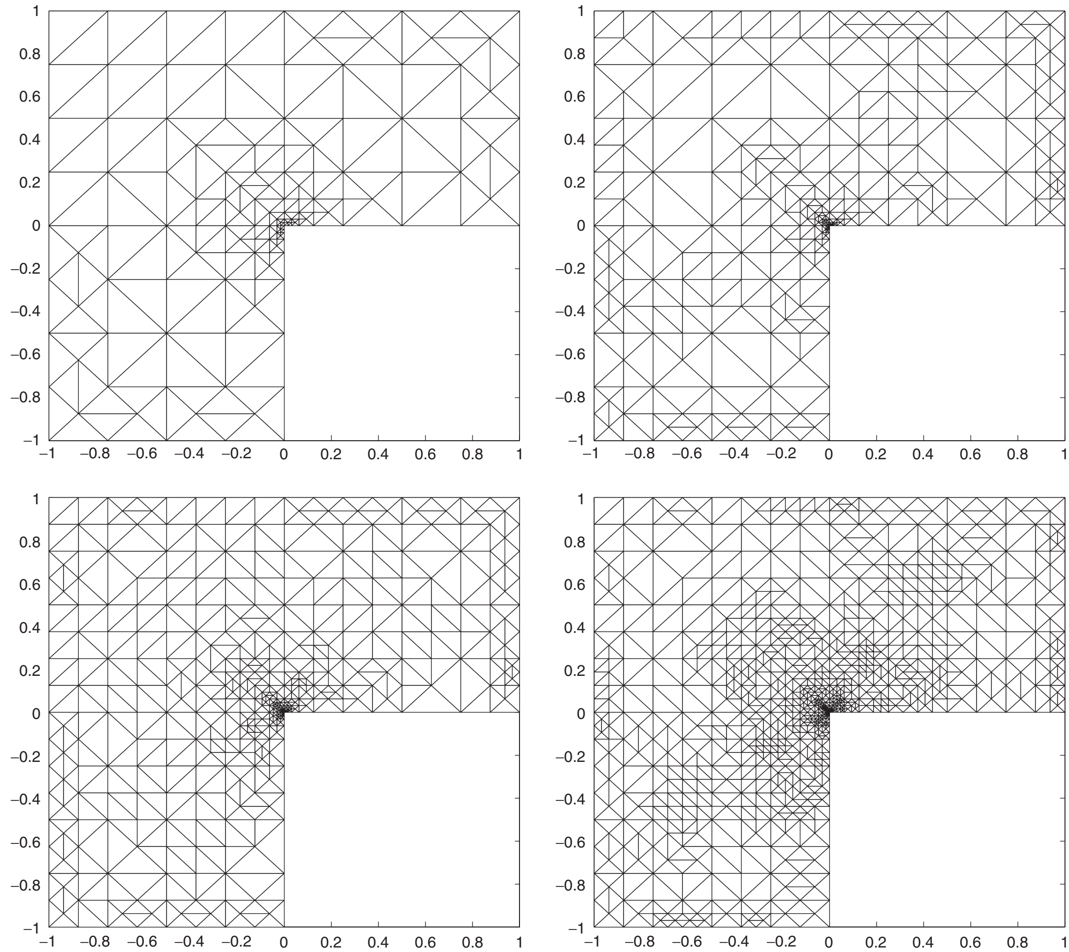

Figure 3. Adapted intermediate meshes with 4043, 8679, 13 642 and 28 569 degrees of freedom, respectively (Example 1).

On the other hand, Figures 3 and 4 show some intermediate meshes obtained with the adaptive refinement. In particular, the meshes in Figure 4 are shown on the corresponding colour plots for the $[L^2(T)]^{2\times 2}$-norm of the stresses $\boldsymbol{\sigma}_h$ on each $T \in \mathcal{T}_h$. As expected, the method is able to recognize all the singularities of the solution pair $(\mathbf{u}, p)$. Indeed, the meshes are highly refined around the origin in Example 1. Similarly, the corresponding adaptive algorithm is able to recognize the region with large stresses in Example 2. We also notice here that the refinement identifies a thin band on a interior neighbourhood of the boundary Γ_{D}, which corresponds to the flat behaviour of the displacement $\mathbf{u}$ caused by the power 2 in the exponent of the exponential function.

Summarizing, the examples presented in this chapter strongly demonstrate that the adaptive algorithm is much more efficient than a uniform discretization procedure when solving the dual-mixed finite element scheme (24).

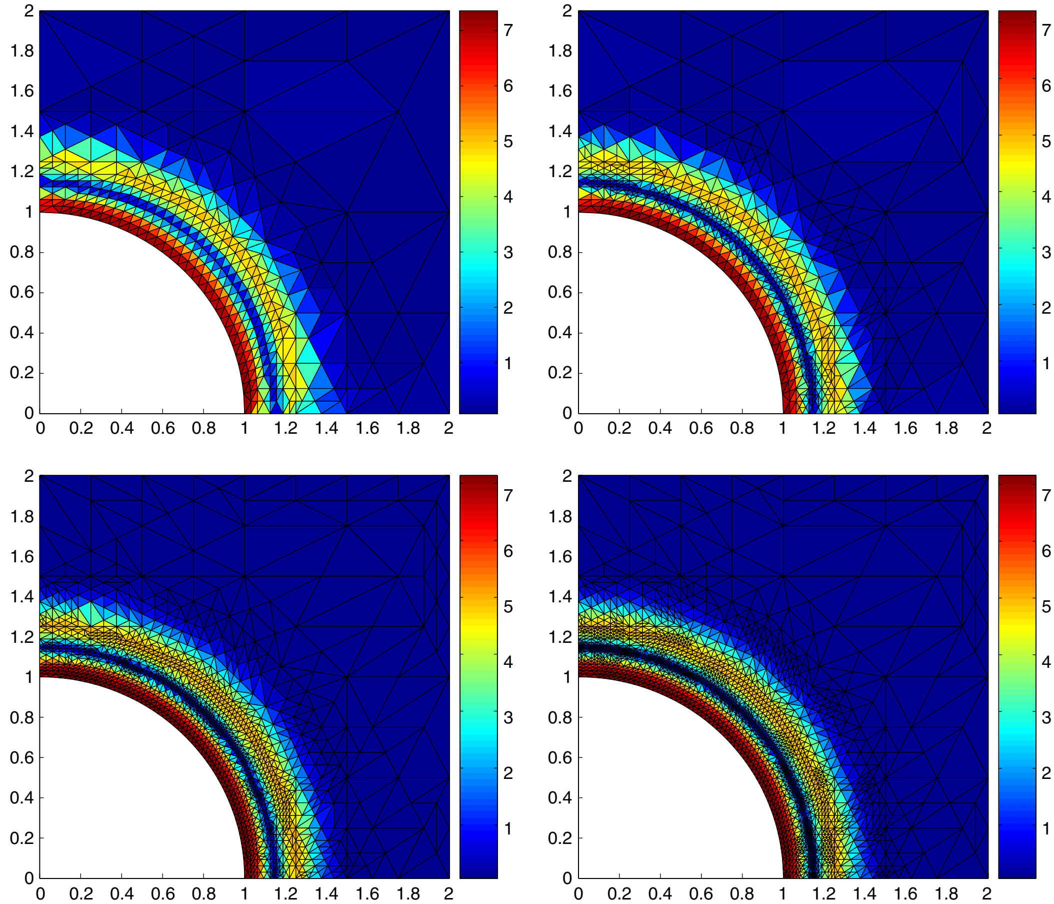

Figure 4. Adapted intermediate meshes with 10 704, 19 068, 46 203 and 67 564 degrees of freedom, respectively (Example 2).

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